Control under uncertainty

Some tricks that really help !

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Introduction

Feedback is always about **uncertainty**

- otherwise, use open-loop optimal control
- typical tools $\rightarrow$ High gain control, observer

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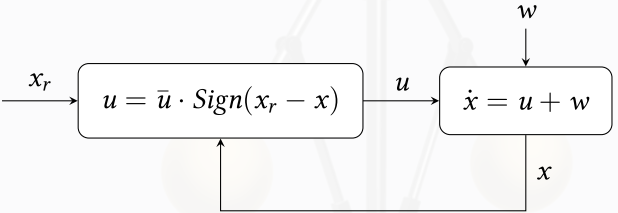
Handling uncertainty by **high-gain** feedback



Let $\epsilon := x_r - x$

If $\bar{u} \geq |w| + \delta$ **then** $\frac{d}{dt}|\epsilon|^2 \leq -\delta|\epsilon|$ $\forall$ constant x_r

Regardless of *w*! $\Rightarrow$ **Kill uncertainty by high-gain feedback**

$\rightarrow$ Higher order sliding modes, Back-stepping, Lyapunov design

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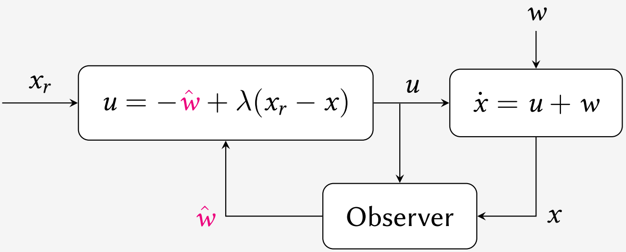
Handling uncertainty via **observer** design



Extend the model

$$\dot{x} = u + w$$
$$\dot{w} = 0$$

Design an observer

$$\dot{\hat{x}} = u + \hat{w} + L_1(x - \hat{x})$$
$$\dot{\hat{w}} = L_2(x - \hat{x})$$

Cancel uncertainty

$$u = -\hat{w} + \lambda(x_r - x)$$

$\Rightarrow$ **Observe & compensate**

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- → Standard approaches.
- need to be reminded / often forgotten
- To investigate before complex solutions

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Feedback is always about **uncertainty**

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- → STANDARD APPROACHES.
- need to be reminded / often forgotten
- To investigate before complex solutions

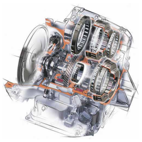
What is beyond and **why**?

- Problems where standard approaches do not apply
- New paradigms → new tools
- Stochastic MPC/DP, Randomized algorithms, Probabilistic certification
- Ex: Energy management, Biological systems

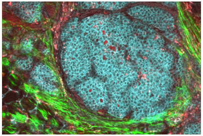
This talk

- Introduce concepts and tools through **use-cases**

Automated manual transmission



Combined therapy of cancer



EV charging station

Propofol-Based Anesthesia

PV power plants

microgrids

Partners

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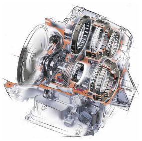

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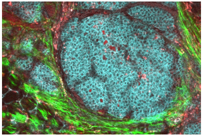

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- Realistic solutions / non trivial problems
- Set of tools / non dogmatic POV
- Easy to remember take-home messages
- Open questions

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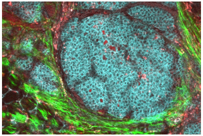

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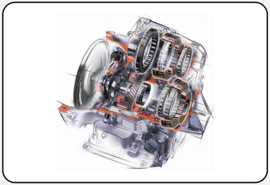
- Personal non exhaustive overview !
- http://www.mazenalamir.fr

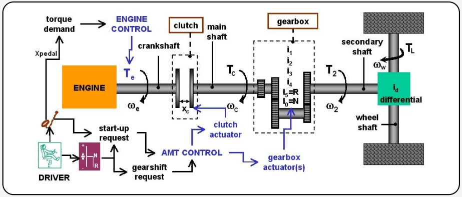
Case study 1

Automated manual transmission



Problem statement

AMT



Control Objective

- Smooth clutch $\omega_{sl} = \omega_e - \omega_c \to 0$
- Transparency
  (torque $\leftrightarrow$ Pedal Position)
  $\omega_e^{ref} = max\left\{\omega_e^0, \mathcal{T}^{-1}\left(T_e^d(X_{pedal}, \omega_e)\right)\right\}$

To summarize

Multivariable constrained tracking / sampling period = 1 ms.

Uncertainties

Inertias, load torque, clutch characteristics, local controllers behavior, road profile...

M. Alamir, A. Murilo, R. Amari, P. Tona, R. Furhapter and P. Ortner, On the use of parameterized NMPC in real-time automotive control. In Automotive Model Predictive Control. L. Del Re (Ed). Lecture Notes in Control and Information Sciences, Springer-Verlag, (2009)

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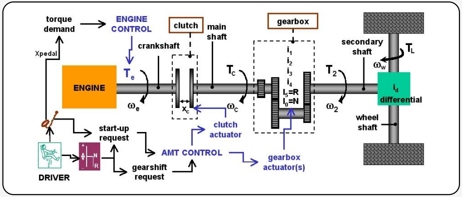

$$\mathcal{J}_e\dot{\omega}_e = T_e - T_c(x_c)$$

$$[\mathcal{J}_c + \mathcal{J}_{eq}(i_g, i_d)]\dot{\omega}_c = T_c(x_c) -$$

$$-\frac{1}{i_g\, i_d}\Big[k_{tw}\theta_{cw} + \beta_{tw}\Big(\frac{\omega_c}{i_g\, i_d} - \omega_w\Big)\Big]$$

$$\mathcal{J}_w\dot{\omega}_w = k_{tw}\theta_{cw} + \beta_{tw}\Big(\frac{\omega_c}{i_g\, i_d} - \omega_w\Big) - T_L(\omega_w)$$

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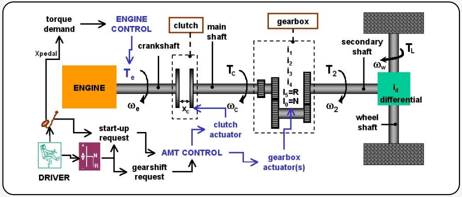

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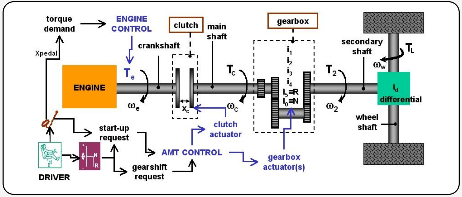

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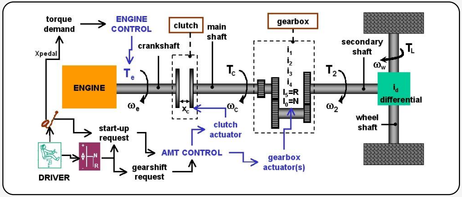

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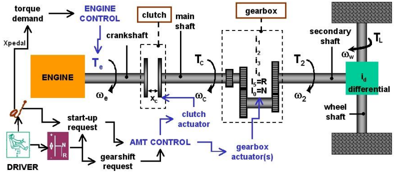

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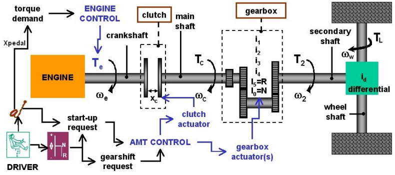

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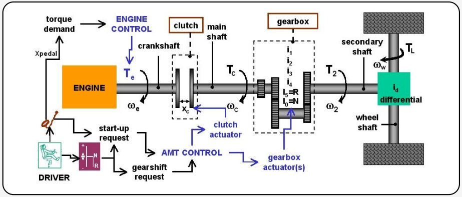

$$\mathcal{J}_e\dot{\omega}_e = u_1 - u_2 - \delta_e$$
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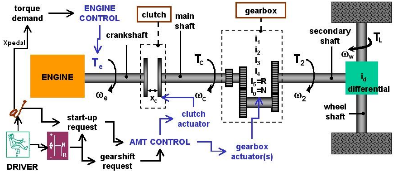

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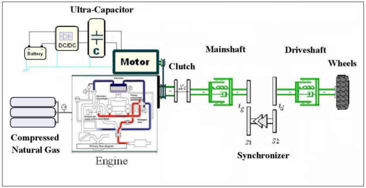
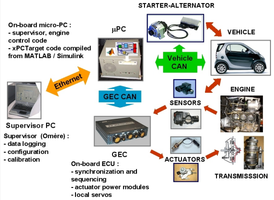
$\to$ Observer-based parameterized MPC

M. Alamir, A. Murilo, R. Amari, P. Tona, R. Furhapter and P. Ortner, On the use of parameterized NMPC in real-time automotive control. In Automotive Model Predictive Control. L. Del Re (Ed). Lecture Notes in Control and Information Sciences, Springer-Verlag, (2009)

The IFPEN SMART demo architecture



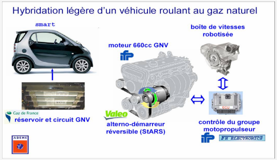
The mild-Hybrid powertrain of the VEHGAN demo-car



VEHGAN on-board Control System

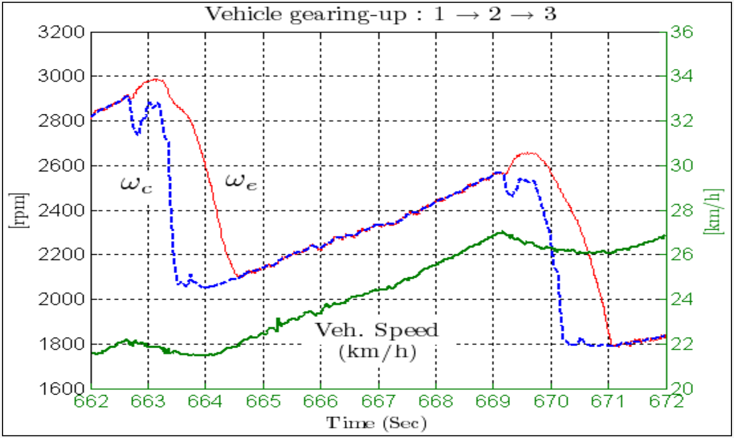
R. Amari (IFPEN)

P. Tona (IFPEN)



Demo SMART car

Experimental Results



Parameterized NMPC - Gearing up

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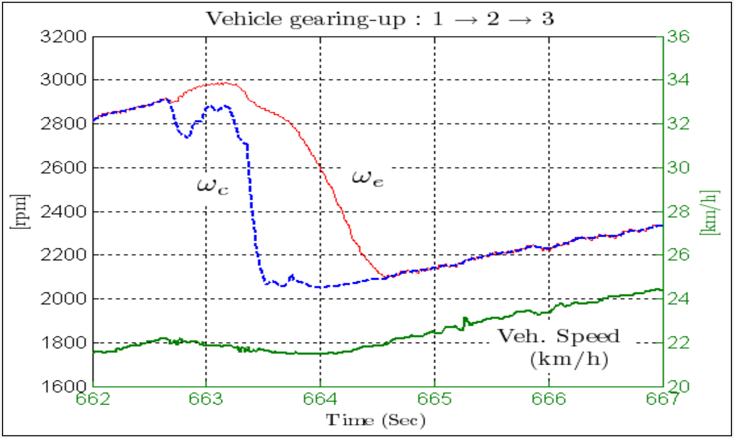

Parameterized NMPC - Gearing up, 1 → 2

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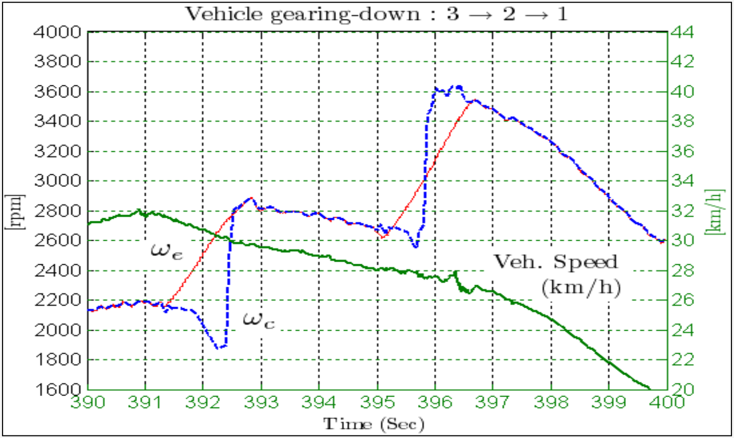
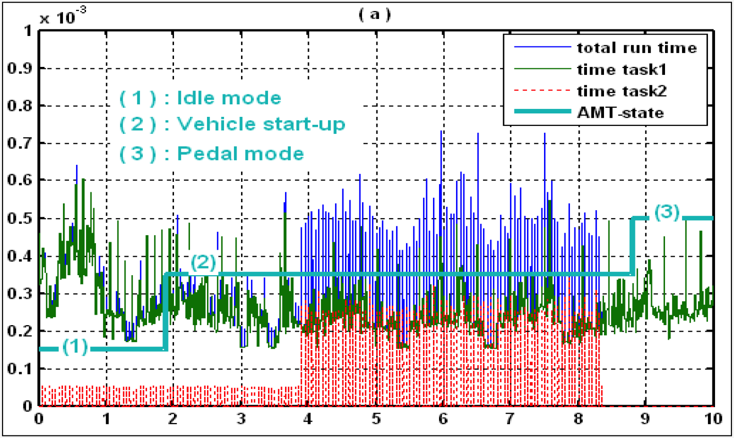


Parameterized NMPC - Gearing down

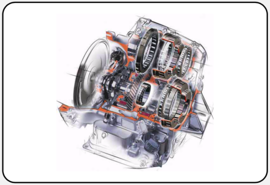
Experimental Results



Parameterized NMPC - Computation time (in 2006 ‼)

Case study 1

Automated manual transmission

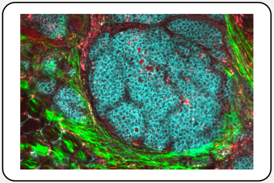


Take away

- Simplify
- Gather **uncertainties** into observable blocks
- Design **observer**-based control
- $\rightarrow$ formulate a **solvable problem**

Case study 2

Combined therapy of cancer



Problem statement

$$\dot{x}_1 = ax_1(1 - bx_1) - c_1 x_4 x_1 - k_3 x_3 x_1$$
$$\dot{x}_2 = -\delta x_2 - k_2 x_3 x_2 + s_2$$
$$\dot{x}_3 = -\gamma_0 x_3 + u_2$$
$$\dot{x}_4 = g\frac{x_1}{h + x_1} x_4 - r x_4 - p_0 x_4 x_1 - k_1 x_4 x_3 + s_1 u_1$$
$$\dot{x}_5 = u_1$$
$$\dot{x}_6 = u_2$$

x_1 tumor cells population
x_2 circulating lymphocytes population
x_3 chemotherapy drug concentration
x_4 effector immune cells population
u_1 injection rate of immune cells
u_2 injection rate of chemotherapy

M. Alamir. On Probabilistic Certification of Combined Cancer Therapies Using Strongly Uncertain Models. **Journal of Theoretical Biology**, Elsevier, Vol. 384, pages 59-69. (2015).

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$$
p := \begin{bmatrix} a \\ b \\ c_1 \\ k_3 \\ \delta \\ k_2 \\ s_2 \\ g \\ h \\ r \\ p_0 \\ k_1 \\ s_1 \end{bmatrix} \in \mathbb{R}^{13}
$$

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Large set of highly uncertain parameters

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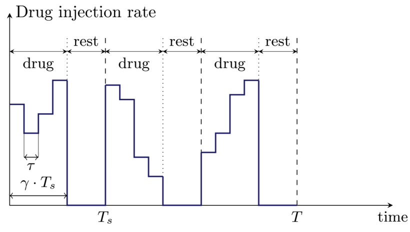
Time constrained protocol



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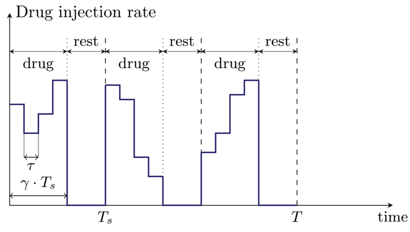
$$\dot{x}_5 = u_1$$

$$\dot{x}_6 = u_2$$

Time constrained protocol



- x_1 tumor cells population
- x_2 circulating lymphocytes population
- x_3 chemotherapy drug concentration
- x_4 effector immune cells population
- u_1 injection rate of immune cells
- u_2 injection rate of chemotherapy

Control objective

- Contract the tumor $x_1(T) \le \gamma_c x_1(0)$
- Monitor health $x_2(t) \ge C_{min}$
- Cost 1: Minimizing drug usage
- Cost 2: Minimizing hospitalization

Problem statement

$$\dot{x}_1 = ax_1(1 - bx_1) - c_1 x_4 x_1 - k_3 x_3 x_1$$
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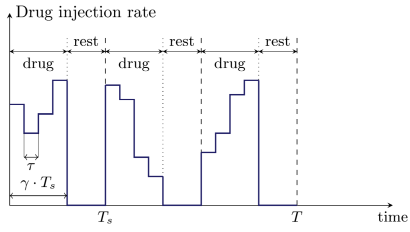

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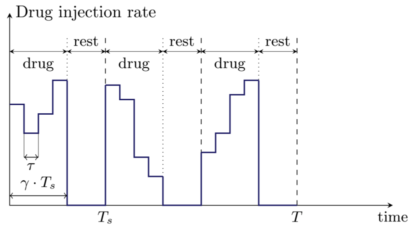

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Cost $\rightarrow \mathcal{J}$

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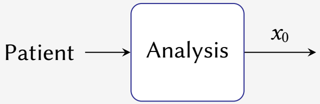

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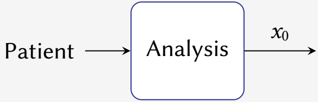
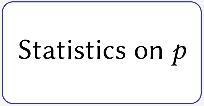
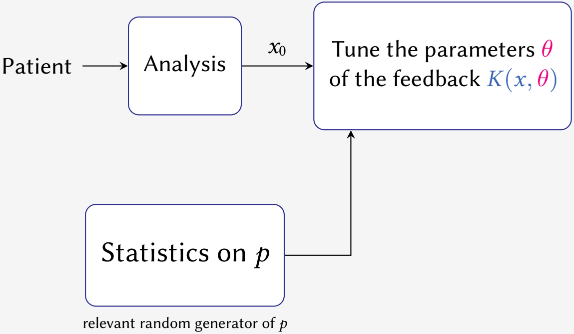




relevant random generator of p

Problem statement



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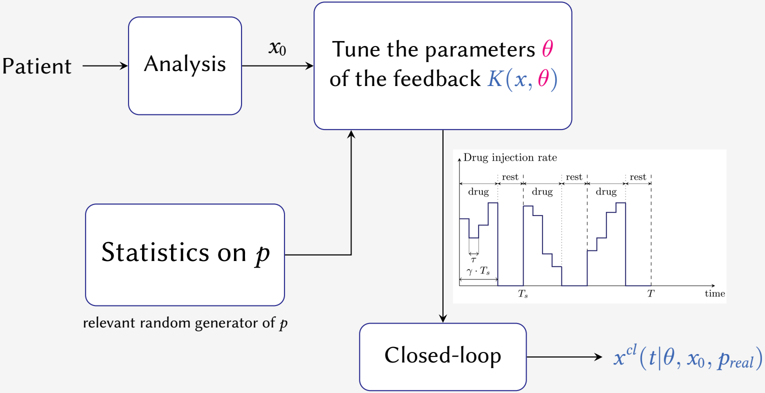

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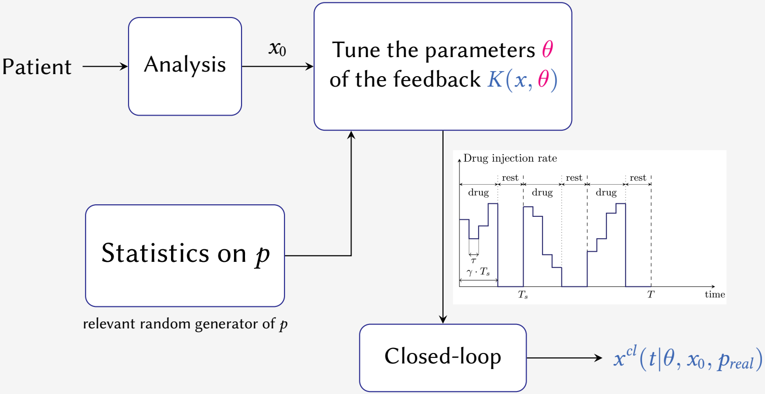

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Find **optimal** θ s.t, it can be certified with a confidence $> 1 - \delta$ that

$$\Pr_{\mathcal{P}}\Big\{ g^{cl}(\theta, p) > 0 \Big\} \leq \eta$$

Typical values $\eta = 10^{-2}$, $\delta = 10^{-3}$

Recalls on probabilistic certification

Probabilistic Certification

A general and **tractable** framework to face high uncertainties and still have something to **guarantee** with an explicit (hopefully high) **probability**.

Probabilistic certification as a relaxed formulation

Optimal Robust constraints satisfaction problem

$$\min_{\theta \in \Theta} \mathcal{J}(\theta) \qquad \text{s.t} \qquad (\forall p \in \mathbb{P}) \quad g(\theta, p) \leq 0$$

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1. Extremely **hard** to solve
2. Uselessly **conservative**
   - worst case analysis
   - even extremely rare bad configurations are accommodated for

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Relaxed formulations $\Rightarrow$ Probabilistic certification

Define the constraints violation indicator:

$$I(\theta, p) := \begin{cases} 0 & \text{if } g(\theta, p) \leq 0 \\ 1 & \text{otherwise (constraints are violated)} \end{cases}$$

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For any given m, N must be sufficiently high to get

- ✓ $N \geq \dfrac{m}{\eta}$
- ✓ $\Pr\Big\{R_2(N) \cap (!R_1)\Big\} \leq \delta$

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What value for N?

When the admissible set Θ is discrete with cardinality n_Θ

$$N \geq \frac{1}{\eta}\left(m + \ln(\frac{n_\Theta}{\delta}) + \left(2m\ln(\frac{n_\Theta}{\delta})\right)^{1/2}\right)$$

more formulas are available (Alamo et al. IEEE-TAC 2009)

T. Alamo

m	positive integer ($m = 1, 5, 10$ etc.)
n_Θ	card(Θ)
η	Precision
δ	Confidence parameter

Values of N for $\delta = 0.01$, $m = 1$

n_Θ	$\eta = 0.1$	$\eta = 0.05$	$\eta = 0.01$	$\eta = 0.001$
1	132	264	1317	13164
5	154	308	1536	15354
10	163	326	1628	16280
100	193	386	1930	19299
1000	223	445	2225	22249
10000	252	503	2515	25148

NOTA

- ✓ dim(p) does not matter !!!!
- ✓ $(n_\Theta, \delta) \rightarrow$ logarithmically !!

Alamo et al. Randomized strategies for probabilistic solutions of uncertain feasibility and optimization problems. IEEE TAC, 2009.

Probabilistic certification as a relaxed formulation

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Relaxed formulations ⇒ Probabilistic certification

Probabilistic certification as a relaxed formulation

optimal design with probabilistic certification

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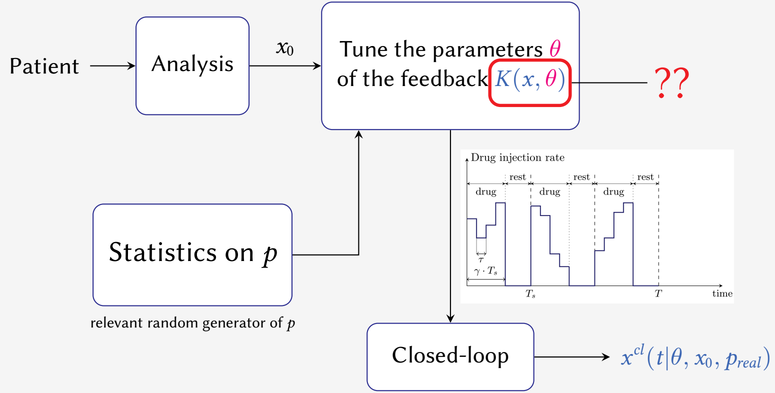

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Definition of $K(x,\theta)$: Chemotherapy design

$$
\begin{aligned}
\dot{x}_1 &= ax_1(1-bx_1) - c_1x_4x_1 - k_3x_3x_1 \\
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$$\dot{x}_2 \geq \mu_2(\beta C_{min} - x_2) \; ; \; \beta > 1$$

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therefore

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u_2 = \min\left\{\bar{u}_2, \gamma_0 x_3^{max}(x, \beta), \frac{D_2 - x_6}{x_7}\right\}
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$$u_2 = \min\left\{\bar{u}_2, \gamma_0 x_3^{max}(x,\beta), \frac{D_2 - x_6}{x_7}\right\}$$

$$\boldsymbol{\theta} := [\boldsymbol{\beta} \quad \boldsymbol{D_1} \quad \boldsymbol{D_2} \quad \ldots]$$

Application

Application / cancer problem:

- Precision $\eta = 10^{-2}$
- Confidence $\delta = 10^{-2}$
- $m = 1$
- $n_\Theta = 576$
- $T = 60$ days

$\rightarrow N = 2155$

$\rightarrow$ Max # simulations $= 1,241,280$

$\rightarrow$ Max computation time = 90 sec

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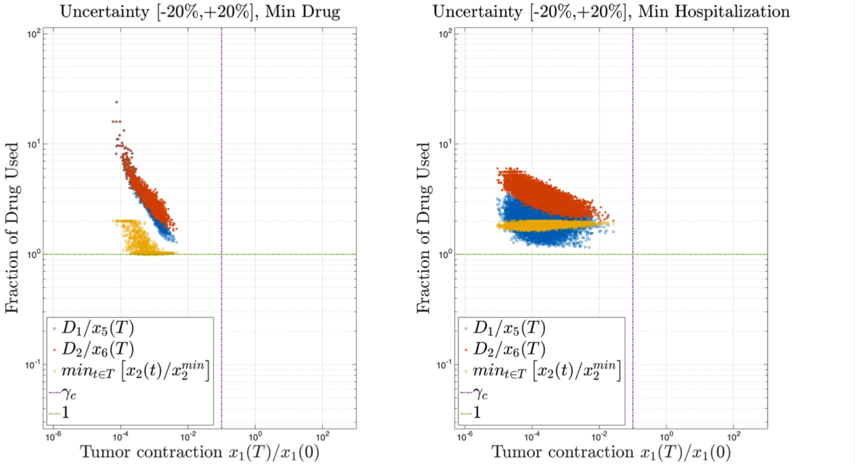
Uncertainties	Min drug	Min Hospitalization
$[-10\%, +10\%]$	$\begin{pmatrix} \beta = 1.05 \\ r = 0.25 \\ \alpha = 0.5 \\ \gamma = 0.8 \\ N_T = 4 \\ \underline{\mathbf{d = 0.25}} \end{pmatrix}$	$\begin{pmatrix} \beta = 1.05 \\ r = 0.5 \\ \alpha = 0.8 \\ \underline{\boldsymbol{\gamma} = \mathbf{0.3}} \\ N_T = 4 \\ d = 0.75 \end{pmatrix}$
$[-20\%, +20\%]$	$\begin{pmatrix} \beta = 1.05 \\ r = 0.25 \\ \alpha = 0.5 \\ \gamma = 0.8 \\ N_T = 4 \\ \underline{\mathbf{d = 0.5}} \end{pmatrix}$	$\begin{pmatrix} \beta = 2 \\ r = 0.5 \\ \alpha = 0.5 \\ \underline{\boldsymbol{\gamma} = \mathbf{0.3}} \\ N_T = 4 \\ d = 0.75 \end{pmatrix}$
$[-40\%, +80\%]$	$\begin{pmatrix} \beta = 2 \\ r = 0.05 \\ \alpha = 0.5 \\ \gamma = 0.8 \\ N_T = 6 \\ \underline{\mathbf{d = 0.5}} \end{pmatrix}$	$\begin{pmatrix} \beta = 2 \\ r = 0.25 \\ \alpha = 0.5 \\ \underline{\boldsymbol{\gamma} = \mathbf{0.5}} \\ N_T = 6 \\ d = 1 \end{pmatrix}$

Validation on $10 \times N = 21550$ random samples



(b) Uncertainty $[-20\%, +20\%]$: (left): case where drug is minimized (right) case hospitalization duration is minimized.

M. Alamir. On Probabilistic Certification of Combined Cancer Therapies Using Strongly Uncertain Models. **Journal of Theoretical Biology**, Elsevier, Vol. 384, pages 59-69. (2015).

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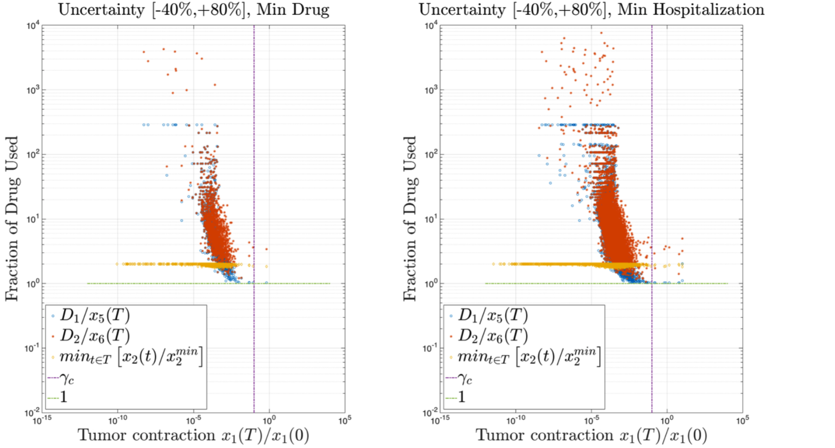

(a) Uncertainty $[-40\%, +80\%]$ (left): case where drug is minimized (right) case hospitalization duration is minimized.

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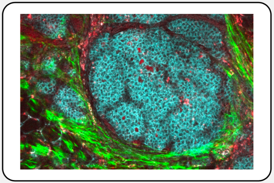
Case study 2

Combined therapy of cancer



Take away

- Design **parameterized** feedback
- Tune the parameters by **randomized optimization**

Ingredients of probabilistic certification

$K(x,\theta)$ Control parametrization

$g(\theta,p)$ Constraints

$\mathcal{J}(\theta)$ cost function

$\mathcal{P}$ Statistics on p to fire relevant N samples $p^{(i)}$

$$\min_{\theta\in\Theta} \mathcal{J}(\theta) \qquad \text{s.t} \qquad \frac{1}{N}\sum_{i=1}^{N}\big[I(\theta,p^{(i)})\big] \le \eta$$

Case study 3

Optimal operation of EV charging station

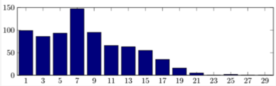
e-Car / Euref Campus, Berlin, Germany

Peter Pflaum

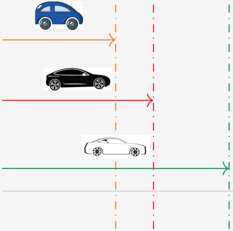
Yacine Lamoudi

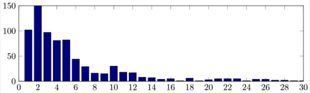
P. Pflaum, M. Alamir and M. Y. Lamoudi. Probabilistic energy management strategy for electric vehicle charging station using randomized algorithms. **IEEE Transactions on Control Systems Technology**, Vol. 26, Issue 3, 2018

Problem Statement: Statistics



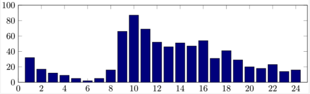
E_{req}: Required energy [kWh]





t_{park}: Parking time [h]

t_{arr}: arrival times



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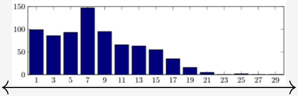

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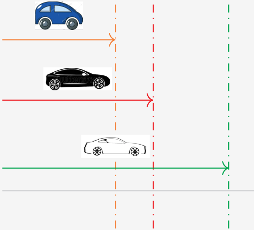

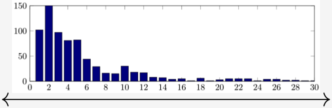

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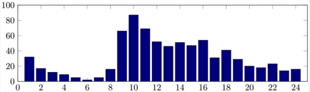
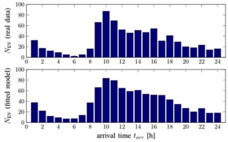
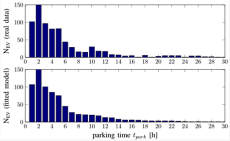
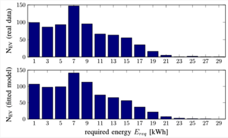


These statistics can be identified:







Ingredients of probabilistic certification

$K(x, \theta)$ Control parametrization

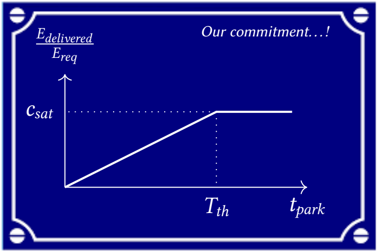
$g(\theta, p)$ Constraints

$\mathcal{J}(\theta)$ cost function

$\mathcal{P}$ Statistics on p to fire relevant N samples $p^{(i)}$

$$\min_{\theta \in \Theta} \mathcal{J}(\theta) \qquad \text{s.t} \qquad \frac{1}{N}\sum_{i=1}^{N}\big[I(\theta, p^{(i)})\big] \leq \eta$$

Problem Statement: Quality of service (QoS)



Certify that

- $\left(t_{park} \geq T_{th} = 3\text{h}\right) \quad \Rightarrow \quad \left(\frac{E_{delivered}}{E_{req}} \geq c_{sat} = 0.9\right)$
- Otherwise $\quad \frac{E_{delivered}}{E_{req}} \geq \left[\frac{t_{park}}{T_{th}}\right] \cdot c_{sat}$

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Problem statement: Decision Variables

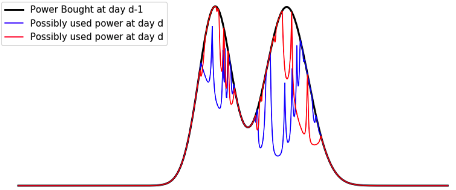
Market protocol

- at day $d-1$: The EV Charging station **buys a power profile P_{grid}**
- at day d: The power P_{EVSC} effectively used by the station **must** satisfy:

$$P_{EVSC}(t) \leq P_{grid}(t)$$

Compute an *optimal power profile $\boldsymbol{P}_{grid}$*

Problem statement: Decision Variables



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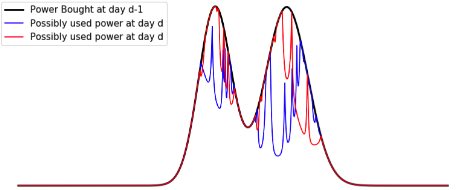

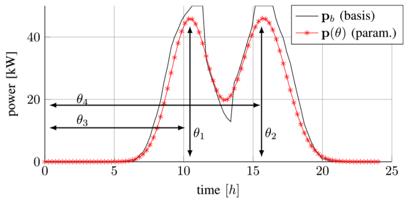

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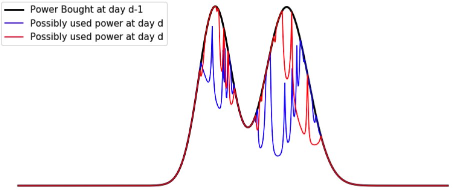

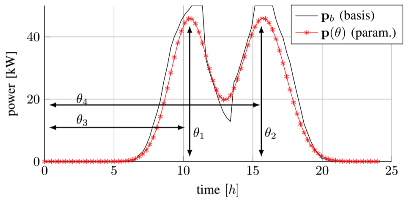

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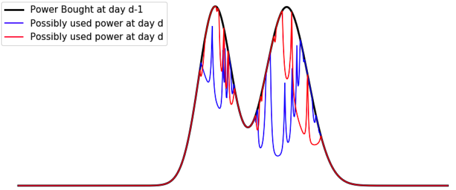

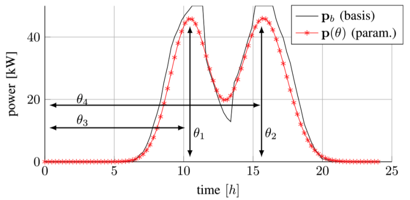

$$\min_{\theta\in\Theta} \int_0^{24} \alpha(t) P_{grid}^{(\theta)}(t)\,dt$$

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Control parameterization

Given $\boldsymbol{P}_{grid}^{(\theta)}$

At each instant t

1. Try to distribute the power uniformly under $\sum P_i \leq P_{grid}^{(\theta)}$
2. If there is remaining power, distribute among EV with lowest SOCs

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Results

Parameters

$\eta = 0.05$ (accuracy)

$\delta = 0.05$ (confidence)

$n_\Theta = 432$ (# design values)

$m = 20$ (# unsuccessful scenarios)

$N_{cp} = 50$ (# charging points)

Required number of scenarios

$$N = \frac{1}{\eta}\left[\frac{e}{e-1}\right]\left[\ln\frac{n_\Theta}{\delta} + m\right] = 920$$

Alamo et al. Randomized methods for the design of uncertain systems: sample complexity and sequential algorithms. **Automatica**, Vol. 52, pp. 1468-1473, 2015.

Upper bound / CPU time

$$\underbrace{t_{single}}_{=63\ msec} \times N \times n_\Theta = 42 min$$

Intel(R) Core(TM) i7-3540M CPU @ 3.00 GHz, 16,0 Go RAM)

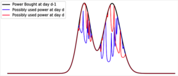
Peter Pflaum

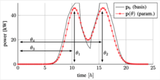
Energy management strategies for smart grids, PhD, Univ. Grenoble Alpes, 2017.

Pflaum et al. Probabilistic energy management strategy for electric vehicle charging station using randomized algorithms. **IEEE Transactions on Control Systems Technology**, Vol. 26, Issue 3, 2018

Results: Euref Campus (Berlin)

Problem statement: Decision Variables





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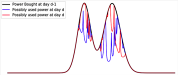

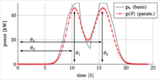

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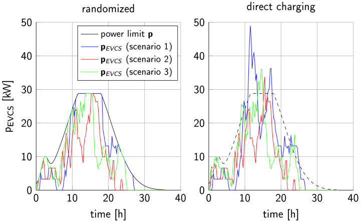
Euref Campus, Berlin



Source: P. Pflaum PhD. 2017.

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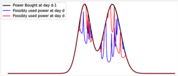

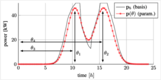

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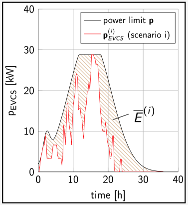

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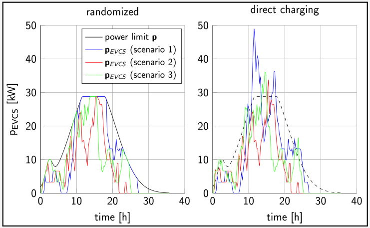


Source: P. Pflaum PhD. 2017.

Average over estimation of power needs

Sensitivity of $\overline{E}$ w.r.t. N_{CP} and η:

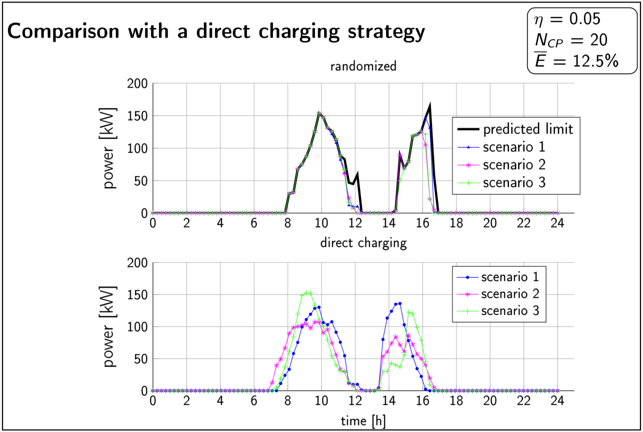
		η = 0.05	0.1	0.15	0.2
N_{CP}	25	58.4	56.3	54.2	51.7
	50	**37.5**	34.1	32.9	31.2
	100	30.6	28.8	27.1	25.1

η probability of QoS violation
N_{CP} number of charging points

Source: P. Pflaum PhD. 2017.

Results: Company Site - (Simulation)

Comparison with direct charging in a a company-like context



Source: P. Pflaum PhD. 2017.

Case study 3

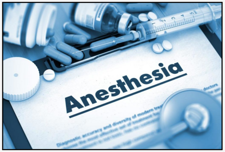
Optimal operation of EV charging station

e-Car / Euref Campus, Berlin, Germany

Take away

- Choose **Carefully** your d.o.f (θ)
- Identify relevant statistics of uncertainties
- Choose *simple* **low-level** decision rules
- Tune by **probabilistic certification**

Similar studies involving probabilistic certification



M. Alamir et al. Feedback law with probabilistic certification for propofol-based control of BIS during anesthesia. International **Journal of Robust and Nonlinear Control**. Volume 28, Issue 18, pages 6254-6266, 2018.

P. Pflaum et al. Battery Sizing for PV power plants under regulations using randomized algorithms. **Renewable Energy (Elsevier)**, Vol 113, pages 596-607, 2017

Ch. Dedouit et al. Certification under uncertainties of control methods for multi-source elevators. In **Advances in intelligent systems and computing series, Springer**, vol. 557, 2017

Stochastic optimal/predictive/DP control

✓ **Avoid a priori choice** of the feedback law's structure

✓ Anticipate **long-term** consequences of uncertainties.
[$\rightarrow$ Avoid non recoverable errors induced by short-sight early actions]

✗ Needs mathematical models!

✗ **Computationally expensive**:

- On-line (MPC: Model Predictive Control)
- Off-line (SDP: Stochastic Dynamic Programming.)

✓ **Second life** $\leftarrow$ Free NLP Solvers, GPU, Machine Learning-Deep Learning, Docker.

Bertsekas, D. P. Dynamic programming and optimal control. 4th Edition, 2017.
Mesbah, A. Stochastic Model Predictive Control with active uncertainties learning. Annual Review Control, 2018.

Formulation

Problem description

Model $x^+ = f(x, u, p)$

Constraints $g(x, p) \leq 0, \quad u \in \mathbb{U}$

Statistics
$$\mu[h] := \int \Pi(p) h(p)\, dp$$
$$\sigma[h] := \int \Pi(p)[h(p) - \mu[h]]^2 dp$$

$$\mathcal{J}(\boldsymbol{u}, x|p) = \sum_{k=0}^{N} \gamma^k \Big[L_0(x_k, u_k) + \rho[g(x_k, u_k)]_+ \Big]$$
$$x_{k+1} = f(x_k, u_k, p),\ x_0 = x$$
$$\boldsymbol{u} = \begin{bmatrix} u_0 & \dots & u_{N-1} \end{bmatrix}$$

$$P(x) : \quad \min_{\boldsymbol{u} \in \mathbb{U}^N} \Big[\mu[\mathcal{J}(\boldsymbol{u}, x|\cdot)] + \alpha \sigma^{0.5}[\mathcal{J}(\boldsymbol{u}, x|\cdot)] \Big]$$

$\boldsymbol{u}^*(x)$ Solution of $P(x)$

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Stochastic MPC

- on-line
- solve $P(x_k)$ at each instant k
- apply $u_0^*(x_k)$ over $[k, k+1]$

- ✓ use deterministic NLP package
- ✓ ACADO, CASADI, PDFMPC,...etc.
- ✓ "*arbitrary*" complex models

- ! No global view
- ! difficult to certify

Formulation

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Stochastic Dynamic Programming

- off-line
- solve the Bellman equation:

$$Q(z) = L(z) + \gamma \min_{v \in \mathbb{U}} [\hat{\mu} + \alpha\hat{\sigma}](Q(f(z, \cdot), v))$$

- ✓ γ-contractive Fixed-point ($\alpha = 0$)
- ✓ easy to code (free ML libraries)
- ✓ SKLEARN, TENSORFLOW,..., etc
- ✓ Highly parallelizable (GPU)
- ✓ Non convex set of control
- ✓ global view of performance

- ! curse of dimensionality
- ! Moderate-size models

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$\boldsymbol{u}^*(x)$ Solution of $P(x)$

In what follows:

- The rationale in using $\alpha > 0$
- Simple example
- Numerical investigation in the case of combined therapy of cancer

Penalizing the variance: the rationale

Worst case:

$$\min_{\boldsymbol{u}}\left[\max_{p\in\mathbb{P}} \mathcal{J}(\boldsymbol{u}\,|p)\right]$$

Expectation:

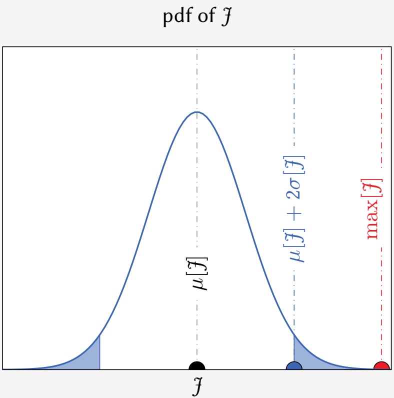
$$\min_{\boldsymbol{u}}\left[\mu[\mathcal{J}(\boldsymbol{u}\,|\cdot)]\right]$$

Worst event with probability $> \eta$:

$$\min_{\boldsymbol{u}}\left[\mu[\mathcal{J}(\boldsymbol{u}\,|\cdot)] + \alpha\sigma^{0.5}[\mathcal{J}(\boldsymbol{u}\,|\cdot)]\right]$$

where
$\alpha = \min\{\alpha \mid \Pr\left[\mathcal{J} > \mu[\mathcal{J}] + \alpha\sigma^{0.5}[\mathcal{J}]\right] \leq \eta\}$

$\rightarrow$ Normal distribution $\mathcal{N}(0,1)$, $\eta = 5\% \rightarrow \alpha \approx 2$



Example

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$$\dot{x}_2 = ux_1 + p\phi_0 \exp(-\lambda(u - u_0)^2)$$

- $u \in [0, 3]$
- Control objective Maximize the production $x_2(T)$
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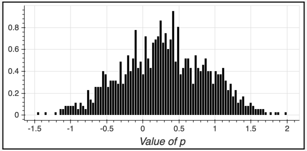

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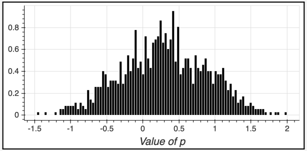

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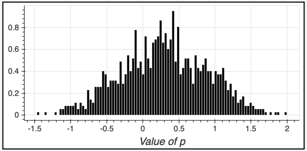

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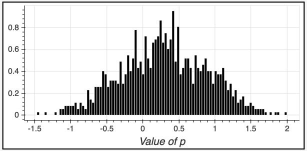

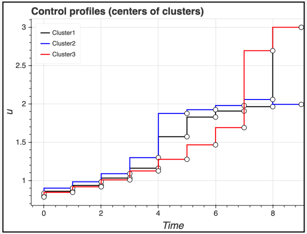

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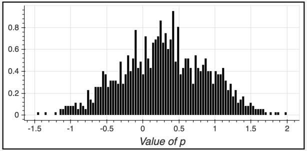

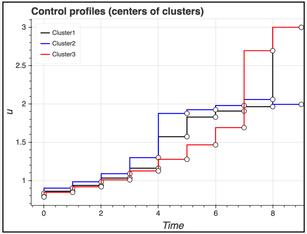

	Mean	Std	Combined (alpha=3)
Cluster #1	0.580505	0.014487	0.537044
Cluster #2	0.601486	0.027406	0.519266
Cluster #3	0.568268	0.000748	0.566025

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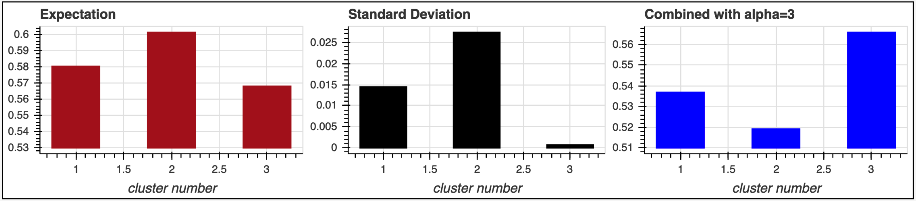

Back to the combined therapy of cancer

Problem description

Model $x^+ = \Phi(x, u)\Psi(p)$

Constraints $g(x, u) := x_2^{min} - x_2 \leq 0, \quad u \in \mathbb{U}$

Statistics $\hat{\mu}[h] := \sum_{s=1}^{n_s} \pi_s h(p^{(s)})$

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$$L_0(x, u) = x_1^2 + \rho_1 u_1 + \rho_2 u_2$$

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$\boldsymbol{u}^*(x)$ Solution of $P(x)$

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Structure of Q

- Take a grid of values of z: $\mathcal{Z} := \{z^{(i)}\}_{i=1}^{9604}$
- Denote by $\boldsymbol{q} := \{q_i\}_{i=1}^{9604}$ the value of Q at $\mathcal{Z}$
- $\hat{Q}$ is the SVM regressor learned from the data $\{\mathcal{Z}, \boldsymbol{q}\}$
- $\rightarrow$ Fixed point iteration on $\boldsymbol{q}$

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Computing $\hat{\mu}$ and $\hat{\sigma}$

- Draw 10^4 samples $\{\Psi^{(i)}\}_{i=1}^{10^4}$ of Ψ
- Extract $n_s = 20$ cluster centers and probability $\{\Psi^{(j)}, \pi_j\}_{j=1}^{n_s}$
- $\hat{\mu} \approx \sum_j \pi_j \cdot Q(f(\Phi(z)\Psi^{(j)}, v)), \qquad \hat{\sigma} \approx \sum_j \pi_j \cdot (Q(f(\Phi(z)\Psi^{(j)}, v) - \hat{\mu})^2$

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Point-wise optimization of v

- Enumerate the four possibilities in $\mathbb{U} := \{0, u_1^{max}\} \times \{0, u_2^{max}\}$

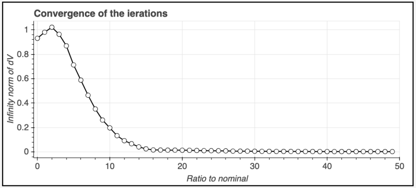
M. Alamir. Explicit approximation of stochastic optimal feedback control for combined therapy of cancer. **arXiv:1905.04937**. 2019.

Approximate SDP for the combined therapy of cancer: methodology for validation

Roadmap

- Using the above methodology, design:
  1. A stochastic DP controller with $\alpha = 0$
  2. A stochastic DP controller with $\alpha = 2$
  3. A nominal controller
- Draw 20000 scenarios (100 initial states $\times$ 200 parameter vectors)
- Simulate each controller on each scenario
- normalize using the results with the nominal controller.
- Show histograms of the results.

Approximate SDP for the combined therapy of cancer: results



Typical convergence of the Fixed-point iteration ($\gamma = 1$)

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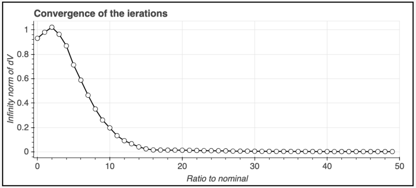

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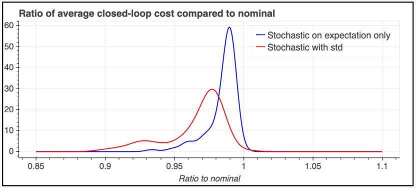

Closed-loop performance comparison on 2×10^4 scenarios

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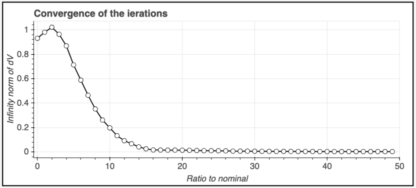

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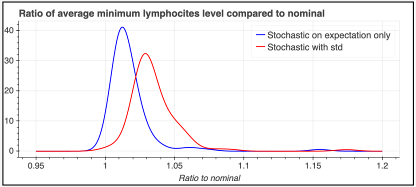
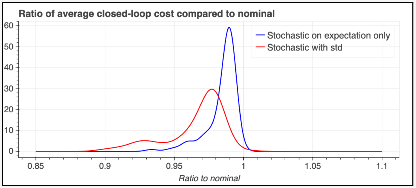


Closed-loop health safety penalty term)



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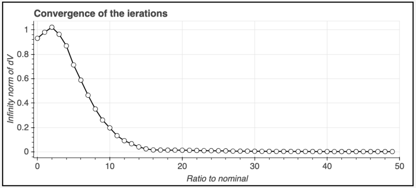

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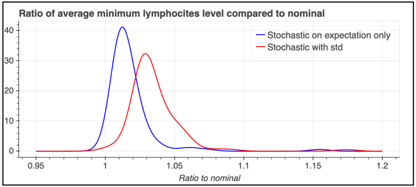

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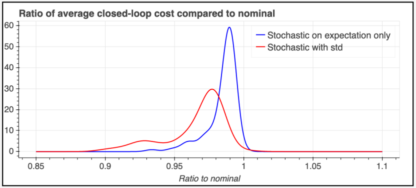

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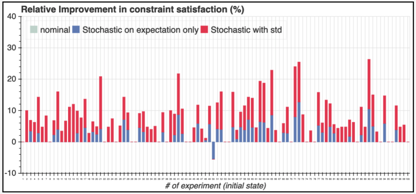

Closed-loop constraints satisfaction statistics

Stochastic optimal control

Take away

- **No ready**-to-use solution!
- **New opportunities**: Free solvers, free ML libraries, GPUs.
- **Sub-optimal** achievable solutions might already be worth

Last take away message …

The story is only starting!

$\exists$ room for imagination