

# ON CONSTRAINED MOVING-HORIZON OBSERVERS IN PRESENCE OF BOUNDED NON GAUSSIAN NOISES

Mazen Alamir<sup>1</sup>

<sup>1</sup>CNRS, Gipsa-Lab, University of Grenoble

CIFA2012, July 4-6, Grenoble-France

## The problem

### STATE ESTIMATION OF LTI SYSTEMS

Construct a dynamic state observer for the LTI-System

$$x^+ = Ax + Bu + Gv$$

$$y = Cx + Hw$$

- ▶  $v$  and  $w$  are bounded noises with known bounds

## The problem

### STATE ESTIMATION OF LTI SYSTEMS

Construct a dynamic state observer for the LTI-System

$$x^+ = Ax + Bu + Gv$$

$$y = Cx + Hw$$

- ▶  $v$  and  $w$  are bounded noises with known bounds
- ▶  $E\{v_i^2\}$  and  $E\{w_i^2\}$  are known **But not**  $E\{v_i v_j\}$  and  $E\{w_i w_j\}$

## The problem

### STATE ESTIMATION OF LTI SYSTEMS

Construct a dynamic state observer for the LTI-System

$$x^+ = Ax + Bu + Gv$$

$$y = Cx + Hw$$

- ▶  $v$  and  $w$  are bounded noises with known bounds
- ▶  $E\{v_i^2\}$  and  $E\{w_i^2\}$  are known **But not**  $E\{v_i v_j\}$  and  $E\{w_i w_j\}$
- ▶  $v$  and  $w$  are **not necessarily independent** ( $E\{v_i w_j\} \neq 0$ )

## The problem

### STATE ESTIMATION OF LTI SYSTEMS

Construct a dynamic state observer for the LTI-System

$$x^+ = Ax + Bu + G\nu$$

$$y = Cx + Hw$$

- ▶  $\nu$  and  $w$  are bounded noises with known bounds
- ▶  $E\{\nu_i^2\}$  and  $E\{w_i^2\}$  are known **But not**  $E\{\nu_i\nu_j\}$  and  $E\{w_iw_j\}$
- ▶  $\nu$  and  $w$  are **not necessarily independent** ( $E\{\nu_iw_j\} \neq 0$ )
- ▶ Estimation is **used in control**. [ $u = K(\hat{x})$ ]

## The problem

### STATE ESTIMATION OF LTI SYSTEMS

Construct a dynamic state observer for the LTI-System

$$x^+ = Ax + Bu + G\nu$$

$$y = Cx + Hw$$

- ▶  $\nu$  and  $w$  are bounded noises with known bounds
- ▶  $E\{\nu_i^2\}$  and  $E\{w_i^2\}$  are known **But not**  $E\{\nu_i\nu_j\}$  and  $E\{w_iw_j\}$
- ▶  $\nu$  and  $w$  are **not necessarily independent** ( $E\{\nu_iw_j\} \neq 0$ )
- ▶ Estimation is **used in control**. [ $u = K(\hat{x})$ ]
- ▶ State is known to satisfy  $Ax \leq B$

## Some bemols on Kalman Filter

- ▶ Constraints

[Rao and Rawlings, Automatica 2001]

## Some bemols on Kalman Filter

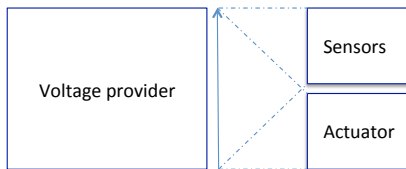
- ▶ Constraints
- ▶ Optimality is questionable

## Some bemols on Kalman Filter

- ▶ Constraints
- ▶ **Optimality is questionable**
  - ▶ Sensor signals are bounded

## Some bemols on Kalman Filter

- ▶ Constraints
- ▶ **Optimality is questionable**
  - ▶ Sensor signals are bounded
  - ▶  $\nu$  and  $w$  may be **coupled**



The feeding voltage noise is simultaneously injected in sensors and actuators

## Some bemols on Kalman Filter

- ▶ Constraints
- ▶ **Optimality is questionable**
  - ▶ Sensor signals are bounded
  - ▶  $\nu$  and  $w$  may be **coupled**

**More generally,**

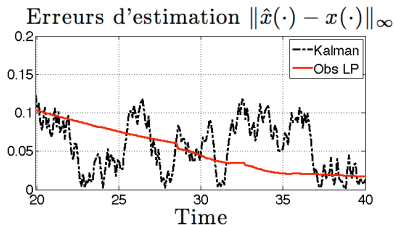
$u = -K\hat{x}$ , if not perfectly measured, the output noise is re-injected in the state equation

## Some bemols on Kalman Filter

- ▶ Constraints
- ▶ Optimality is questionable
  - ▶ Sensor signals are bounded
  - ▶  $\nu$  and  $w$  may be **coupled**
  - ▶  $E\{\nu_i\nu_j\}$ ,  $E\{w_iw_j\}$  are never known [and hence arbitrarily chosen]

## Some bemols on Kalman Filter

- ▶ Constraints
- ▶ **Optimality is questionable**
  - ▶ Sensor signals are bounded
  - ▶  $\nu$  and  $w$  may be **coupled**
  - ▶  $E\{\nu_i \nu_j\}$ ,  $E\{w_i w_j\}$  are never known [and hence arbitrarily chosen]
- ▶ In observer based control, minimizing the variance of  $x - \hat{x}$  is not necessarily the best option

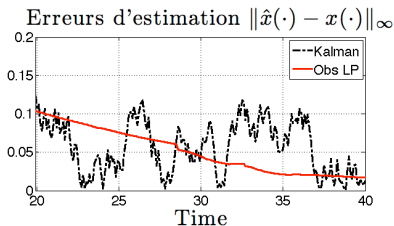


Assuming that the  $L_2$  norm of the Kalman-Filter error is lower, would you **really prefer the black curve** to the red when thinking of sending them to actuators ?

## Some bemols on Kalman Filter

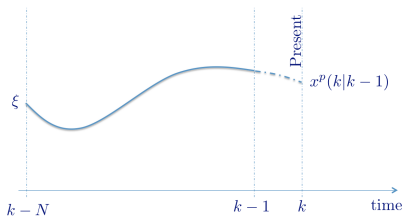
- ▶ Constraints
- ▶ **Optimality is questionable**
  - ▶ Sensor signals are bounded
  - ▶  $\nu$  and  $w$  may be **coupled**
  - ▶  $E\{\nu_i \nu_j\}$ ,  $E\{w_i w_j\}$  are never known [and hence arbitrarily chosen]
- ▶ In observer based control, minimizing the variance of  $x - \hat{x}$  is not necessarily the best option

⇒ Even without the constraint-related issue, there is room for novel thoughts



Assuming that the  $L_2$  norm of the Kalman-Filter error is lower, would you **really prefer the black curve** to the red when thinking of sending them to actuators ?

## Kalman-like approach

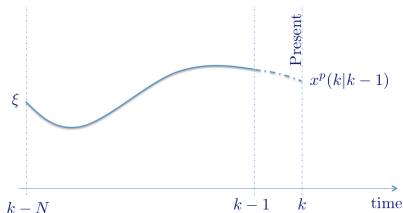


### Solve unconstrained quadratic confidence trade-off

$$\min_{\xi} \left[ \|\hat{x}(k|\xi) - x^p(k|k-1)\|_{Q_s}^2 + \sum_{i=-\infty}^k \|\hat{y}(i|\xi) - y_m(i)\|_{Q_y}^2 \right]$$

- ▶  $Q_s$  and  $Q_y$  depend on  $E\{\nu\nu^T\}$  and  $E\{ww^T\}$
- ▶ Recursive LS implementation  $\leftrightarrow$  Kalman updating rules

## The proposed approach

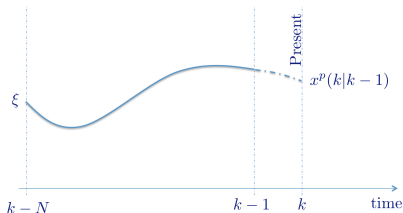


Compute the closest admissible state to the expected one

$$\min_{\eta^{(1)} \geq 0, \eta^{(2)} \geq 0, z = (\xi, \tilde{\nu}, \tilde{w})} \left[ \sum_{i=1}^n (\eta_i^{(1)} + \eta_i^{(2)}) \right] \quad \text{under}$$

- ▶  $y^p(k|z) = y_m(k)$
- ▶  $\hat{x}(k|z) - x^p(k|k-1) = \eta^{(1)} - \eta^{(2)}$
- ▶ Temporal constraints on  $\tilde{\nu}$  and  $\tilde{w}$  [Kolmogorov-Pokhorov]
- ▶ Knowledge-base constraints on the state  $Ax \leq B$

## The proposed approach



Compute the closest admissible state to the expected one

$$\min_{\eta^{(1)} \geq 0, \eta^{(2)} \geq 0, z = (\xi, \tilde{\nu}, \tilde{w})} \left[ \sum_{i=1}^n (\eta_i^{(1)} + \eta_i^{(2)}) \right] \quad \text{under}$$

- ▶  $y^p(k|z) = y_m(k)$
- ▶  $\hat{x}(k|z) - x^p(k|k-1) = \eta^{(1)} - \eta^{(2)}$
- ▶ Temporal constraints on  $\tilde{\nu}$  and  $\tilde{w}$  [Kolmogorov-Pokhorov]
- ▶ Knowledge-base constraints on the state  $Ax \leq B$

## SOME DEFINITIONS &amp; NOTATION

Consider  $N$  random independent variables  $\{V_i\}_{i=1}^N$  satisfying  $E(V_i) = 0$  and denote by  $\sigma_N^2$  their sum of variances:

$$\sigma_N^2 := \sum_{i=1}^N E(V_i^2)$$

Assume that

$$(\forall i \in \{1, \dots, N\}) \quad |V_i| \leq \bar{V}$$

and denote their partial sum by :

$$S_N := \sum_{i=1}^N V_i$$

## KOLMOGOROV-POKHOROV INEQUALITIES [STOUT-74]

The following inequality is satisfied for all  $\epsilon > 0$ :

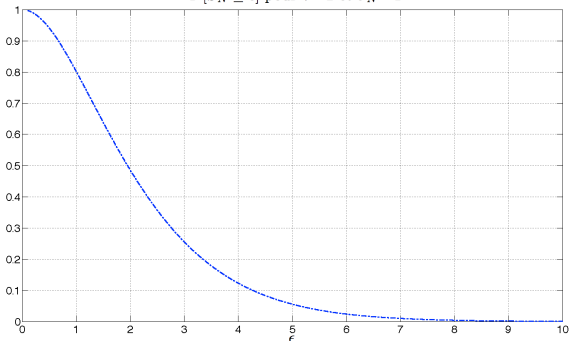
$$Pr[|S_N| \geq \epsilon \sigma_N] \leq \exp\left(-\frac{\epsilon \sigma_N}{2\bar{V}} \cdot \sinh^{-1}\left(\frac{\epsilon \bar{V}}{2\sigma_N}\right)\right)$$

## KOLMOGOROV-POKHOROV INEQUALITIES [STOUT-74]

The following inequality is satisfied for all  $\epsilon > 0$ :

$$Pr[|S_N| \geq \epsilon \sigma_N] \leq \exp\left(-\frac{\epsilon \sigma_N}{2\bar{V}} \cdot \sinh^{-1}\left(\frac{\epsilon \bar{V}}{2\sigma_N}\right)\right)$$

$P[S_N \geq \epsilon]$  pour  $\bar{V}=2$  et  $\sigma_N=1$

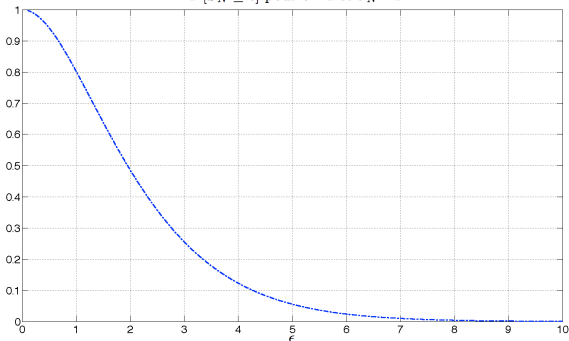


## KOLMOGOROV-POKHOROV INEQUALITIES [STOUT-74]

The following inequality is satisfied for all  $\epsilon > 0$ :

$$Pr[|S_N| \geq \epsilon \sigma_N] \leq \exp\left(-\frac{\epsilon \sigma_N}{2\bar{V}} \cdot \sinh^{-1}\left(\frac{\epsilon \bar{V}}{2\sigma_N}\right)\right) =: \eta$$

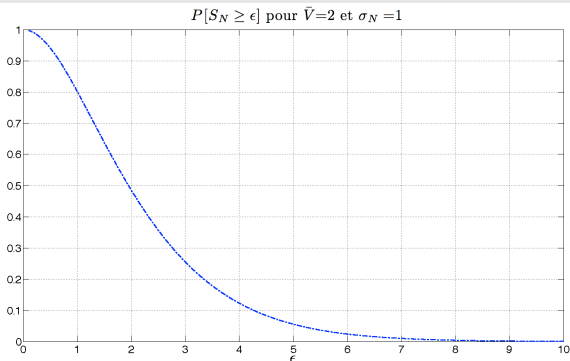
$P[S_N \geq \epsilon]$  pour  $\bar{V}=2$  et  $\sigma_N=1$



## KOLMOGOROV-POKHOROV INEQUALITIES [STOUT-74]

The following inequality is satisfied for all  $\epsilon > 0$ :

$$Pr[|S_N| \geq \epsilon \sigma_N] \leq \Gamma_{(\bar{V}, \sigma_N)}^{-1}(\epsilon) =: \eta$$

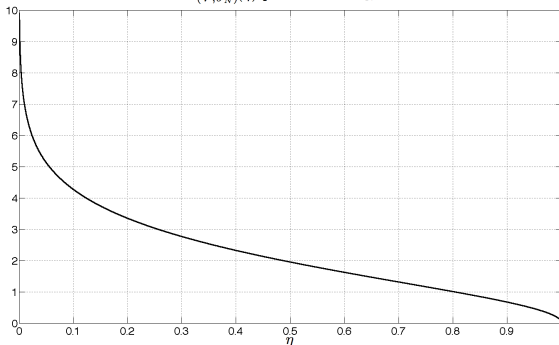


## KOLMOGOROV-POKHOROV INEQUALITIES [STOUT-74]

The following inequality is satisfied for all  $\epsilon > 0$ :

$$Pr[|S_N| \geq \Gamma_{(\bar{V}, \sigma_N)}(\eta) \cdot \sigma_N] \leq \eta$$

$\Gamma_{(\bar{V}, \sigma_N)}(\eta)$  pour  $\bar{V}=2$  et  $\sigma_N=1$



## KOLMOGOROV-POKHOROV INEQUALITIES [STOUT-74]

The following inequality is satisfied for all  $\epsilon > 0$ :

$$Pr[|S_N| \geq \Gamma_{(\bar{v}, \sigma_N)}(\eta) \cdot \sigma_N] \leq \eta$$

For all  $N$ , the following is satisfied with a probability of  $(1-\eta)\%$

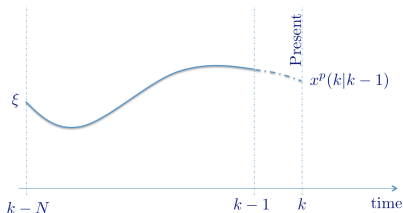
$$\forall s \in \{1, \dots, n_\nu\}$$

$$\left| \sum_{l=0}^{N-1} \nu_s(k+l) \right| \leq \Gamma_{(\bar{\nu}_s, \sqrt{N Q_\nu(s,s)})}(\eta) \cdot \sqrt{N \cdot Q_\nu(s,s)}$$

$$\forall s \in \{1, \dots, n_w\}$$

$$\left| \sum_{l=0}^{N-1} w_s(k+l) \right| \leq \Gamma_{(\bar{w}_s, \sqrt{N Q_w(s,s)})}(\eta) \cdot \sqrt{N \cdot Q_w(s,s)}$$

## The proposed approach



Compute the closest admissible state to the expected one

$$\min_{\eta^{(1)} \geq 0, \eta^{(2)} \geq 0, z = (\xi, \tilde{v}, \tilde{w})} \left[ \sum_{i=1}^n (\eta_i^{(1)} + \eta_i^{(2)}) \right] \quad \text{under}$$

- ▶  $y^p(k|z) = y_m(k)$
- ▶  $\hat{x}(k|z) - x^p(k|k-1) = \eta^{(1)} - \eta^{(2)}$
- ▶ Temporal constraints on  $\tilde{v}$  and  $\tilde{w}$  [Kolmogorov-Pokhorov]
- ▶ Knowledge-base constraints on the state  $Ax \leq B$

Consider the system

$$\dot{x} = A_c x + B_c u + G_c v$$

$$y = Cx + Hw$$

Consider the system

$$\begin{aligned}\dot{x} &= A_c x + B_c u + G_c v \\ y &= Cx + Hw\end{aligned}$$

$$A_c := \begin{pmatrix} -1 & 1 & 0.2 \\ 0 & -1 & 3 \\ 0.2 & 0.6 & -2.7 \end{pmatrix} ; B_c := \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$G_c := \begin{pmatrix} 10 & 0.5 \\ -2 & -10 \\ 10 & 10 \end{pmatrix} ; C := (1 \ 1 \ 1) ; H = 1$$

Consider the system

$$\begin{aligned}\dot{x} &= A_c x + B_c u + G_c v \\ y &= Cx + Hw\end{aligned}$$

- ▶  $\nu_1 := b_1 \cdot \tanh(e_1)$
- ▶  $\nu_2 := b_2 \cdot \tanh(e_2)$
- ▶  $w := b_3 \cdot \tanh(e_3)$
- ▶  $e_i$  white Gaussian noises
- ▶  $(\text{var}(e_1), \text{var}(e_2), \text{var}(e_3)) := (0.5, 0.5, 0.8)$
- ▶

$$\sigma_i^2 := b_i^2 \int_{-\infty}^{\infty} \left[ \frac{e^{-v^2/(2 \cdot \text{var}(e_i))}}{\sqrt{2\pi \cdot \text{var}(e_i)}} \right] \cdot [\tanh(v)]^2 dv$$

- ▶  $(b_1, b_2, b_3) \in \left\{ (0.02, 0.02, 2), (0.2, 0.2, 2) \right\}$

Consider the system

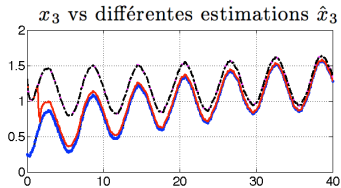
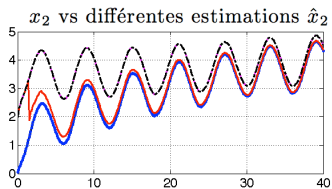
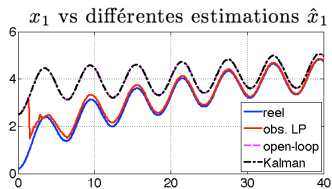
$$\begin{aligned}\dot{x} &= A_c x + B_c u + G_c v \\ y &= Cx + Hw\end{aligned}$$

- ▶ Sampling period  $\tau = 0.1$
- ▶  $N_O = 15$
- ▶ The control is defined by:

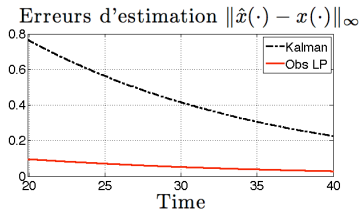
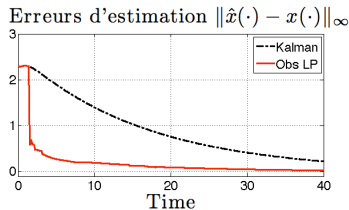
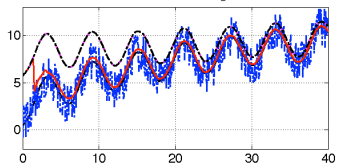
$$u(t) := \frac{2}{5} \sin\left(\frac{2\pi t}{6}\right) + \frac{2}{10} \min\left\{1, \max\{0.4, b\}\right\} \quad (1)$$

- ▶  $b$  is a white noise

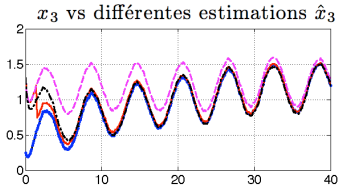
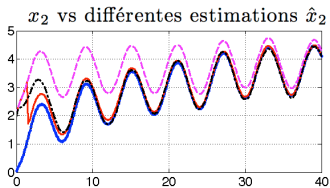
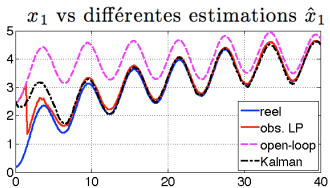
- ▶ Tune Kalman filter according to  $\sigma_i$
- ▶ No way to consider the bounds
- ▶ KP Inequalities threshold  $\eta = 0.1$



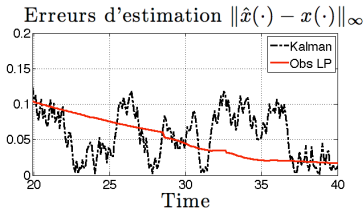
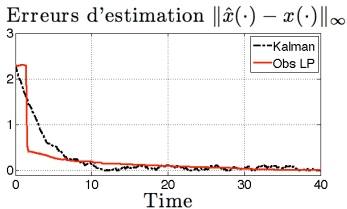
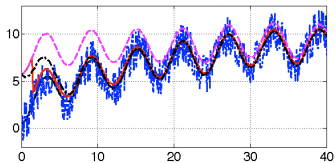
Mesure bruitée  $y$  vs  $C\hat{x}$



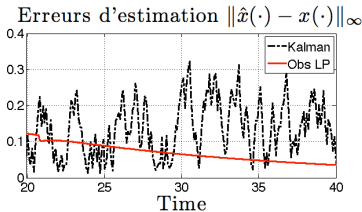
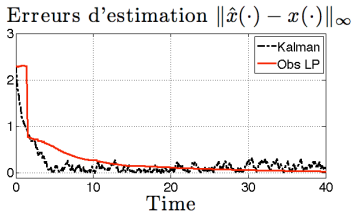
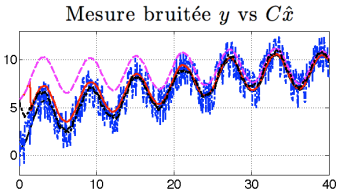
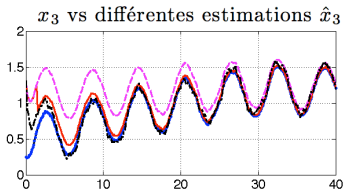
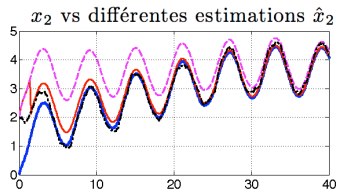
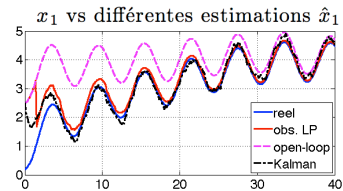
- ▶ → KF too slow (not enough confidence in measurements)
- ▶ Tune Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-5}} \cdot \sigma_3$
- ▶ Increase the confidence in output measurements
- ▶ KP Inequalities threshold  $\eta = 0.1$



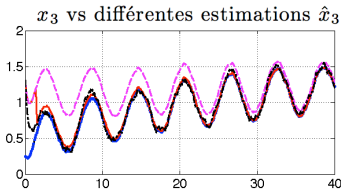
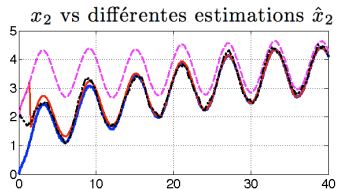
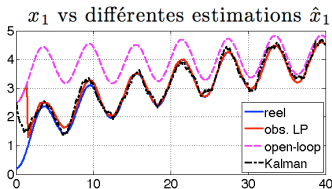
Mesure bruitée  $y$  vs  $C\hat{x}$



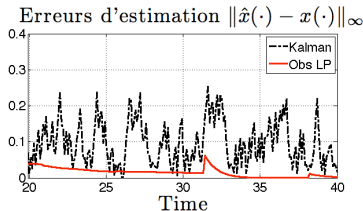
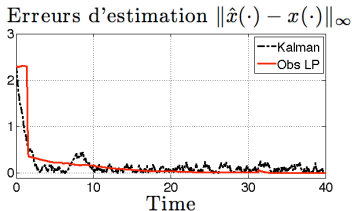
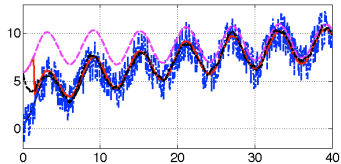
- ▶ KF is  $\rightarrow$  still too slow (not enough confidence in measurements)
- ▶ Tune Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-6}} \cdot \sigma_3$
- ▶ Increase the confidence in output measurements
- ▶ KP Inequalities threshold  $\eta = 0.1$



- ▶ → KF is quite fast but too sensitive to noise
- ▶ Tune Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-6}} \cdot \sigma_3$
- ▶ KP Inequalities threshold increased from  $\eta = 0.1$  to  $\eta = 0.4$

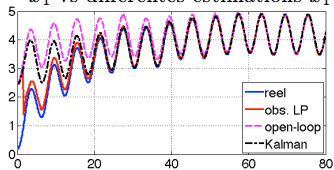


Mesure bruitée  $y$  vs  $C\hat{x}$

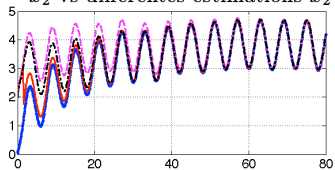


- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-4}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.1$
- ▶ Let's compare the asymptotic behavior ...

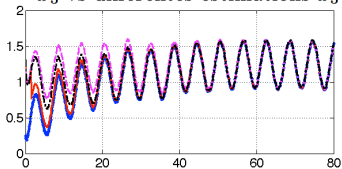
$x_1$  vs différentes estimations  $\hat{x}_1$



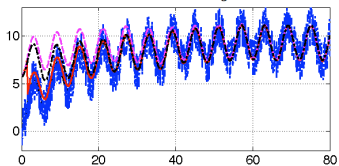
$x_2$  vs différentes estimations  $\hat{x}_2$



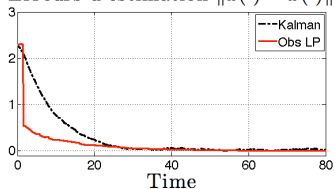
$x_3$  vs différentes estimations  $\hat{x}_3$



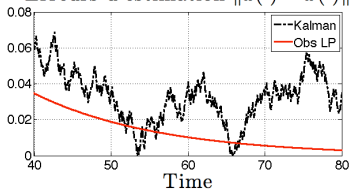
Mesure bruitée  $y$  vs  $C\hat{x}$



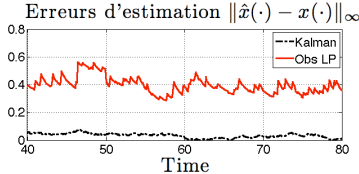
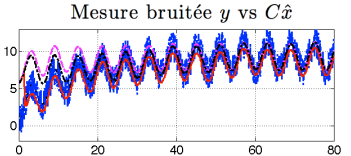
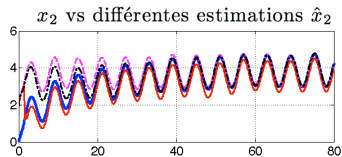
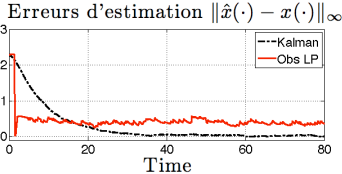
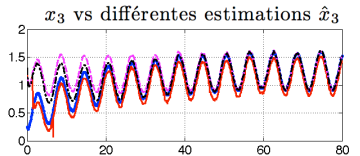
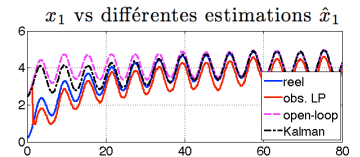
Erreurs d'estimation  $\|\hat{x}(\cdot) - x(\cdot)\|_\infty$



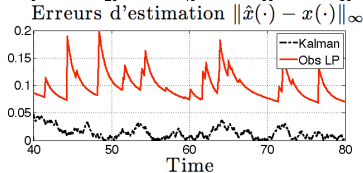
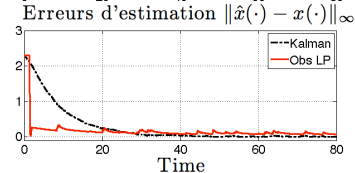
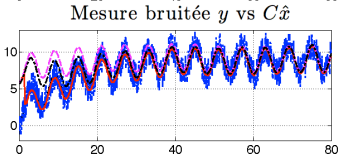
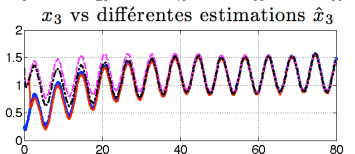
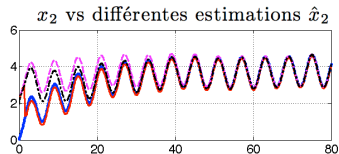
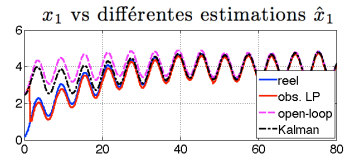
Erreurs d'estimation  $\|\hat{x}(\cdot) - x(\cdot)\|_\infty$



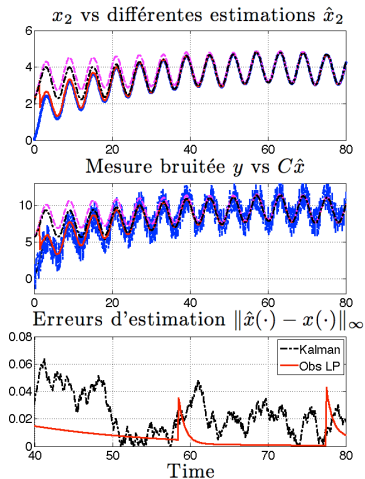
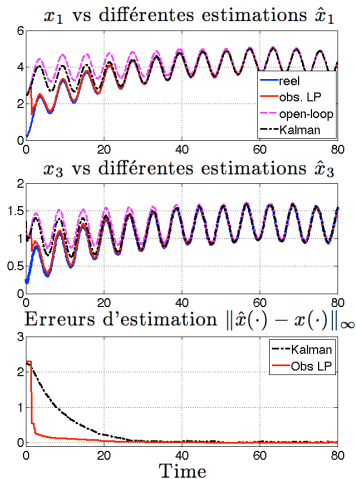
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-4}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.95$
- ▶ Let's compare the asymptotic behavior ...



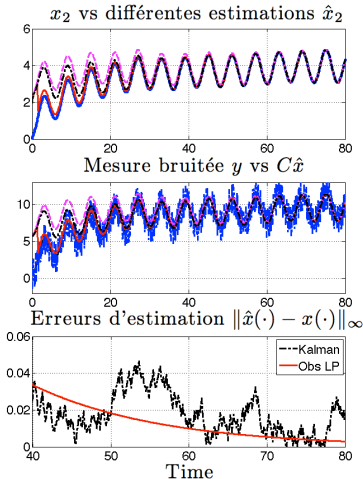
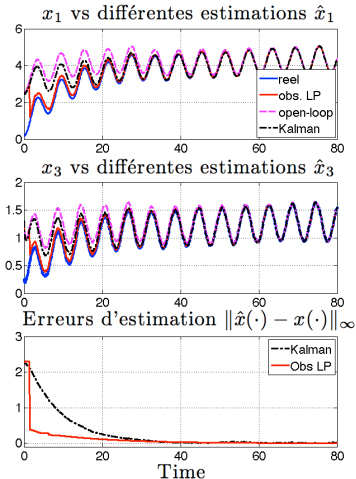
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-4}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.6$
- ▶ Let's compare the asymptotic behavior ...



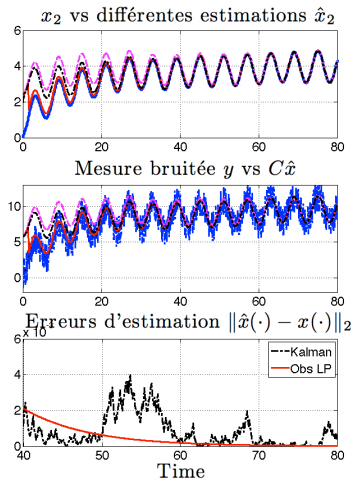
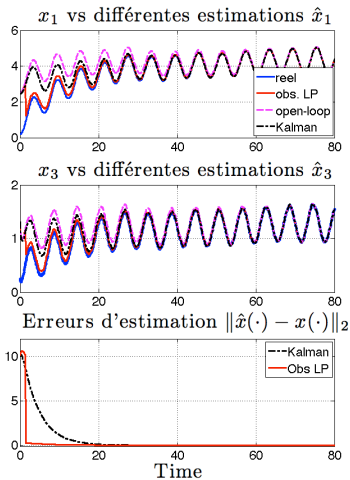
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-4}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.2$
- ▶ Let's compare the asymptotic behavior . . .



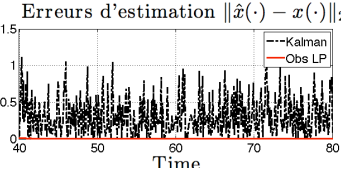
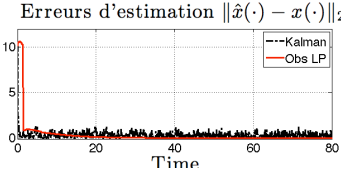
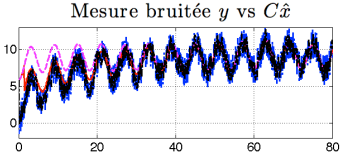
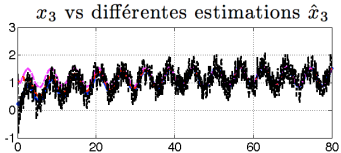
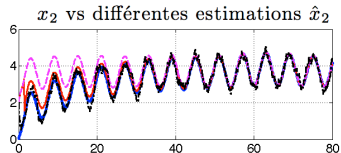
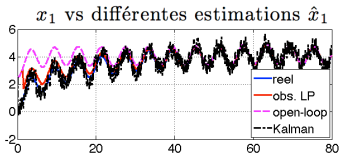
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-4}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.1$
- ▶ Let's compare the asymptotic behavior . . .



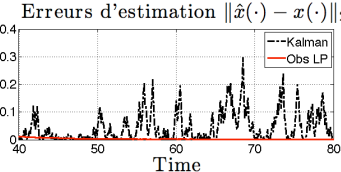
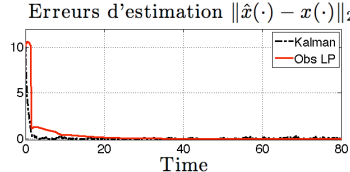
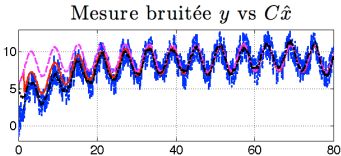
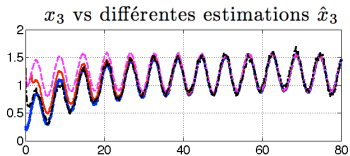
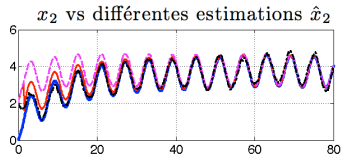
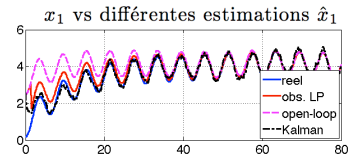
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1$ ,  $\sigma_2$  and  $\sqrt{10^{-4}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.1$
- ▶ Let's compare the asymptotic behavior ...



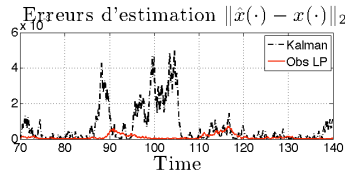
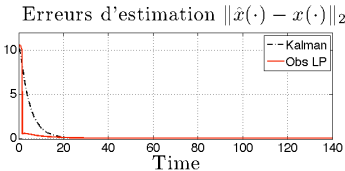
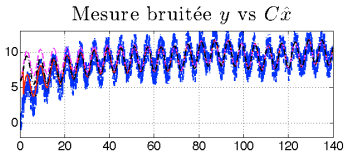
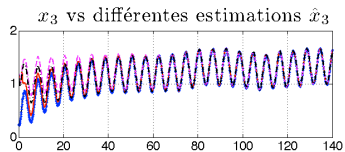
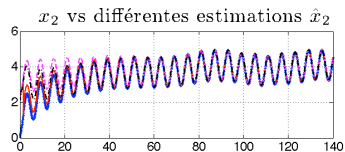
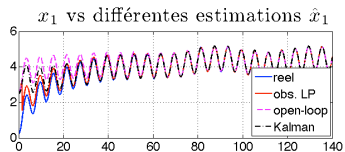
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.2$  and  $\sqrt{10^{-4}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.1$
- ▶ Let's compare the asymptotic behavior ...



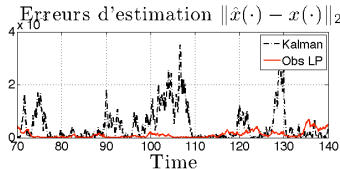
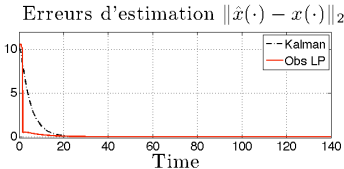
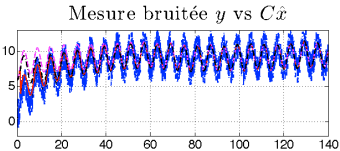
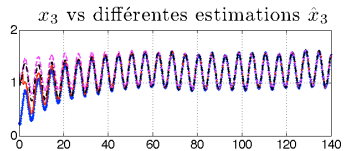
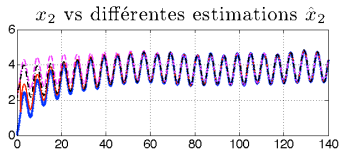
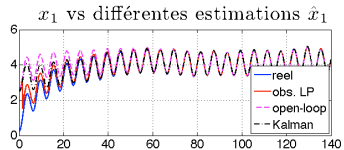
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.2$  and  $\sqrt{10^{-2}} \cdot \sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.1$
- ▶ Let's compare the asymptotic behavior ...



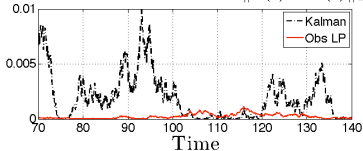
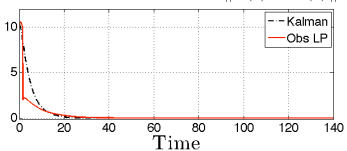
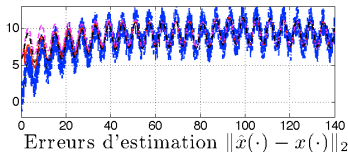
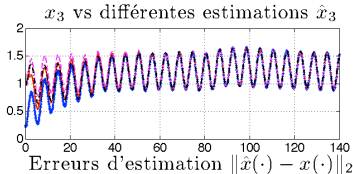
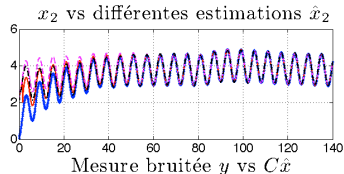
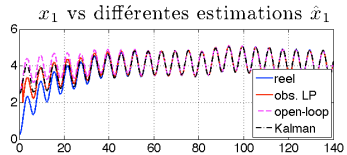
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.2$  and  $\sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.1$
- ▶ Let's compare the asymptotic behavior ...



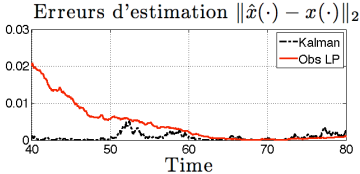
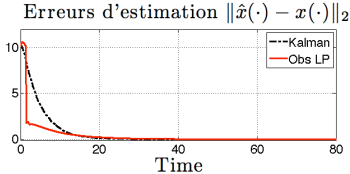
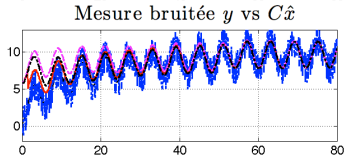
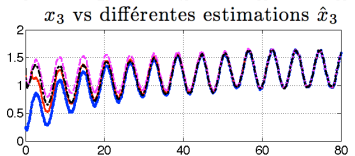
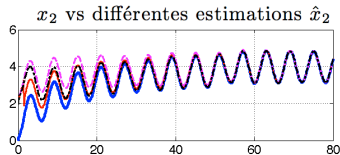
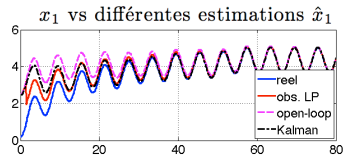
- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.2$  and  $\sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.1$
- ▶ Let's compare the asymptotic behavior . . .



- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.2$  and  $\sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.025$
- ▶ Let's compare the asymptotic behavior . . .



- ▶ Choose a good trade-off for Kalman filter according to  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.2$  and  $\sigma_3$
- ▶ Reset the KP Inequalities threshold to  $\eta = 0.025$
- ▶ Let's compare the asymptotic behavior ...



- ▶ Preliminary work
- ▶ Solution particularly appealing for observer-based control
- ▶ KP-like related curves can be obtained experimentally
- ▶ More investigation is needed regarding
  - ▶ Stability assessment
  - ▶ The impact of observation horizon
  - ▶ The KP inequalities threshold
- ▶ Are the qualitative conclusions general ?