Re-injecting Structure in NMPC Schemes

Application to the constrained stabilization of a snakeboard

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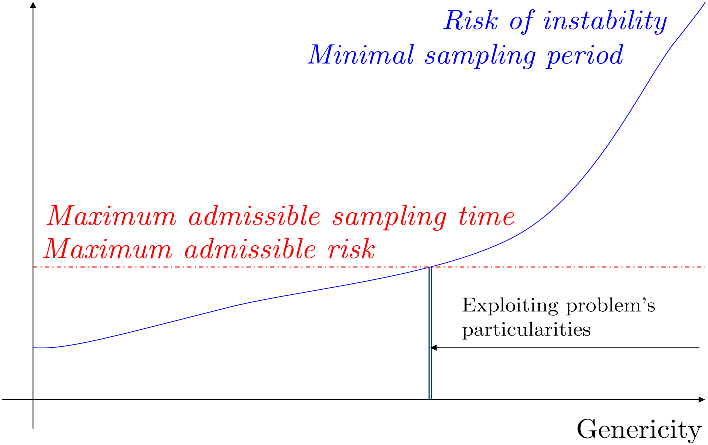
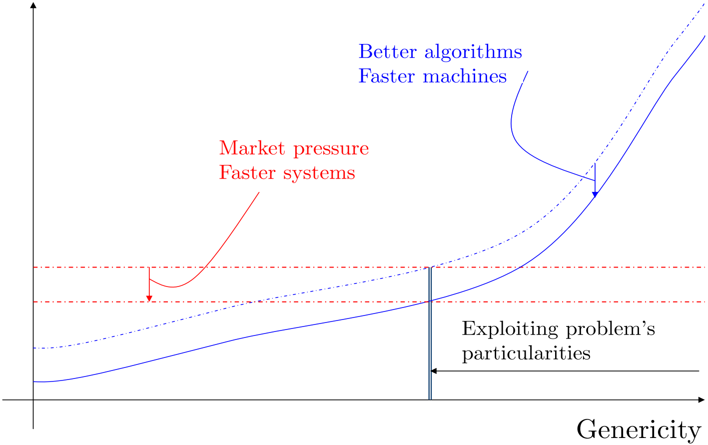
Recall on NMPC

- At instant t, solve a constrained open-loop optimal control problem $J(x(t), u_t, u_{t+1}, \dots, u_{t+N})$
- Apply the first part $\hat{u}_t(x(t))$ over the sampling interval $[t, t+1]$.
- At instant $t+1$, solve $J(x(t+1), u_{t+1}, \dots, u_{t+N+1})$
- Apply $\hat{u}_{t+1}(x(t+1))$ during $[t+1, t+2]$
- etc.
- State feedback $\hat{u}_\cdot(x(\cdot))$

General formulation + General solver

→ "*Plug and Play*" controller ...?


Risk of instability
Minimal sampling period
Maximum admissible sampling time
Maximum admissible risk
Exploiting problem's particularities
Genericity


Better algorithms
Faster machines
Market pressure
Faster systems
Exploiting problem's
particularities
Genericity

This works uses the snakeboard as **an example** to illustrate how the structure of the system can be exploited as far as possible **before** the optimization problem to be used in NMPC-framework is defined in order to obtain an almost **instantaneously tractable** optimization problem.

Satellite in failure mode
(J. of Dyn. Syst. Meas. and Control 2003, J. Optim. in Engineering 2003)

The minimum interception time problem
(Control Engineering practice 2000)

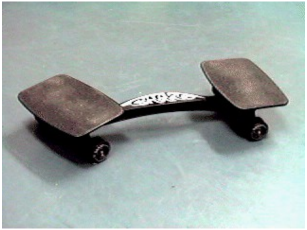
Nonholonomic systems

(J. of Optim. & Appl. 2003 / Automatica 2003)

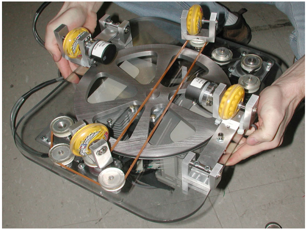
. . .

Outline

The snakeboard



A snakeboard "*without rider*"



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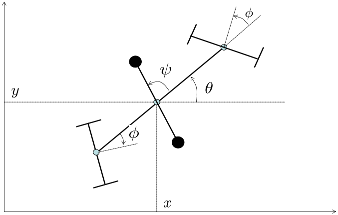


- Conservation of the angular momentum
- No slip at the wheel level
- Regulated variables (x, y, θ)
- Configuration variables (ψ, ϕ)
- Control input (u_ψ, u_ϕ)

Bibliographical notes (1)

(Lewis, A. et al.) "*Nonholonomic Mechanics and locomotion : the snakeboard Example*" IEEE Int. Conf. on Robotics & Automation (1994)

- Original principle of nonholonomic locomotion
- First model
- Analysis of open loop cyclic gaits
- Local controllability except for isolated singular configurations

(Ostrowski, Burdick) "*Controllability Tests for Mechanical Systems with Constraints and Symmetry*" J. of Appl. Math. and Comp. Sc. (1997)

- Achieving controllability proof.

Bibliographical notes (2)

(Ostrowski) "*Steering for a Class of Nonholonomic Systems*" IEEE Trans. on Automatic Contr. (2000)

- Steering trajectories based on small amplitude, short period cyclic inputs.

(Vela, Burdick) "*Control of Biomimetic locomotion via averaging theory*" ICRA (2003)

- Averaging cyclic behavior
- Average stability via state feedback

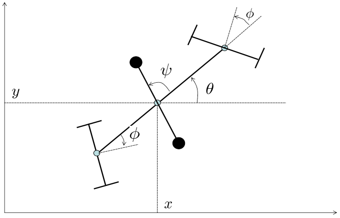
(Iannitti, Lynch) "*Exact Minimum Control Switch Motion Planning for a Snakeboard*" IROS (2003)

- Switching between two vector fields
- System at rest at switching instants
- Minimizing the number of switches
- Connecting sub-curves computed geometrically
- Handling constraints by time scaling
- No feedback.

This work

- Oscillation free solution (at least a priori)
- Constrained state feedback (Path planning+NMPC)
- Path planning $\leftrightarrow$ Solving LP problems
  - Fast computation
  - Easy extension to obstacle avoidance context
- Minimizing spatial excursion in the $(x-y)$ plane
- Assignable maximum number of switches
- Scalar on-line optimization problem for NMPC

$$
\begin{aligned}
\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ \dot{\theta} \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\cos^2\phi \\ 0 \\ \sin(2\phi) \end{pmatrix} \xi \\
\dot{\xi} &= \left(-\frac{J_r}{2m^2}\ddot{\psi} + \dot{\phi}\xi\right) \cdot \tan\phi \\
u_2 &= 2J_w\ddot{\phi} \\
u_1 &= J_r\left[(1 - \frac{J_r}{m^2}\sin^2\phi)\ddot{\psi} + 2\dot{\phi}\xi\cos^2\phi\right]
\end{aligned}
$$



Control task

Transfer the system from $(\bar{x}_0, \bar{y}_0, \theta_0, \xi_0, \phi_0, \psi_0)$ to the desired value $(0, 0, 0, 0, *, *)$ using **feedback** (u_1, u_2) s.t

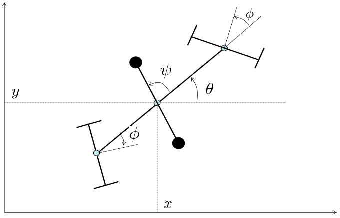
$$u_i(t) \in [-u_i^{max}, +u_i^{max}] \qquad \forall t$$

$$\begin{pmatrix}\dot{\bar{x}}\\ \dot{\bar{y}}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\cos^2\phi\\ 0\\ \sin(2\phi)\end{pmatrix}\xi$$

$$\dot{\xi} = \left(-\frac{J_r}{2m^2}\ddot{\psi} + \dot{\phi}\xi\right)\cdot\tan\phi$$

$$u_2 = 2J_w\ddot{\phi}$$

$$u_1 = J_r\left[(1 - \frac{J_r}{m^2}\sin^2\phi)\ddot{\psi} + 2\dot{\phi}\xi\cos^2\phi\right]$$



Analysis (1)

- Kinematic stage
- Dynamic stage

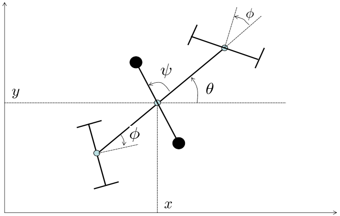
(Classical when using Lie-Group related modelling tools)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\cos^2\phi\\ 0\\ \sin(2\phi)\end{pmatrix}\xi$$

$$\dot{\xi} = \left(-\frac{J_r}{2m^2}\ddot{\psi} + \dot{\phi}\xi\right)\cdot\tan\phi$$

$$u_2 = 2J_w\ddot{\phi}$$

$$u_1 = J_r\left[(1-\frac{J_r}{m^2}\sin^2\phi)\ddot{\psi} + 2\dot{\phi}\xi\cos^2\phi\right]$$



Analysis (2)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \frac{2}{\tan\phi}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\dot{\theta}$$

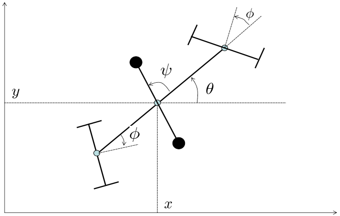
Nonholonomic constraint : Constraint on velocities that is not derivation of constraint on position.

$$\begin{pmatrix}\dot{\bar{x}}\\ \dot{\bar{y}}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\cos^2\phi\\ 0\\ \sin(2\phi)\end{pmatrix}\xi$$

$$\dot{\xi} = \left(-\frac{J_r}{2m^2}\ddot{\psi} + \dot{\phi}\xi\right)\cdot\tan\phi$$

$$u_2 = 2J_w\ddot{\phi}$$

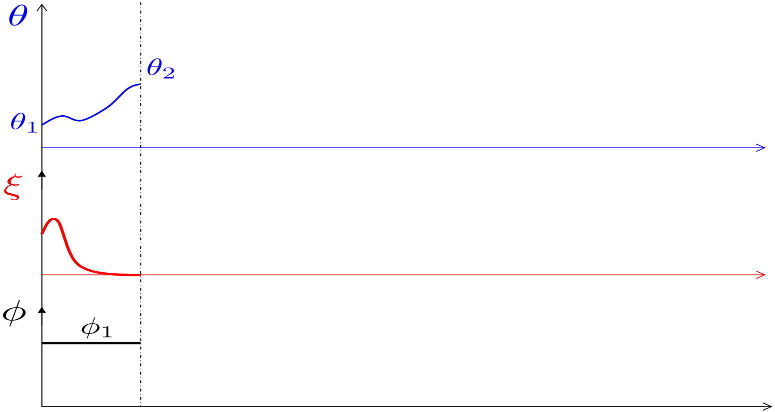
$$u_1 = J_r\left[(1 - \frac{J_r}{m^2}\sin^2\phi)\ddot{\psi} + 2\dot{\phi}\xi\cos^2\phi\right]$$

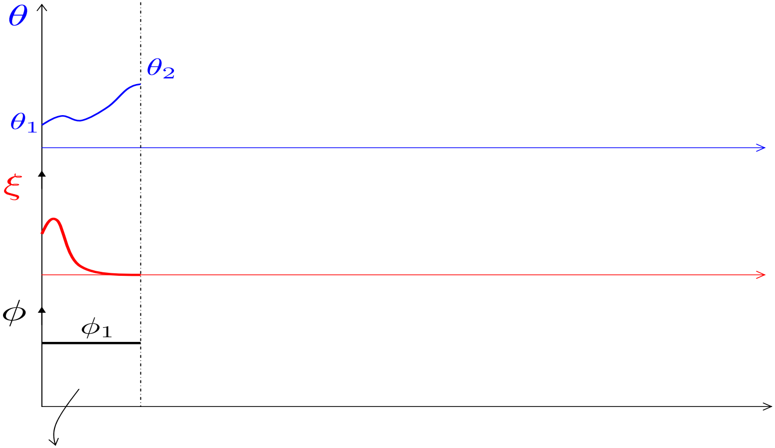


Analysis (2) : Under constant ϕ

$$\begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \frac{2}{\tan\phi}\begin{pmatrix}\Delta(\sin\theta)\\ -\Delta\cos\theta\end{pmatrix}$$

Variations on $(\bar{x},\bar{y})$ are independent on the path used to join two values θ_1 and θ_2.


θ
θ₂
θ₁
ξ
ϕ
ϕ₁



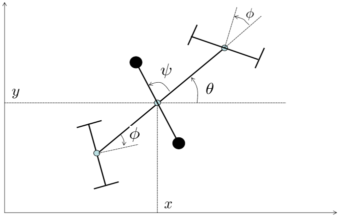
$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\left(F(\theta_1,\theta_2)\right)$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\cos^2\phi\\ 0\\ \sin(2\phi)\end{pmatrix}\xi$$

$$\dot{\xi} = \left(-\frac{J_r}{2m^2}\ddot{\psi} + \dot{\phi}\xi\right)\cdot\tan\phi$$

$$u_2 = 2J_w\ddot{\phi}$$

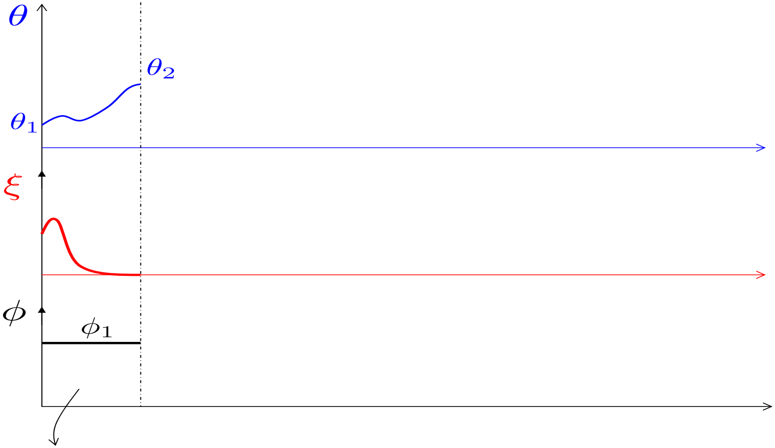
$$u_1 = J_r\left[(1-\frac{J_r}{m^2}\sin^2\phi)\ddot{\psi} + 2\dot{\phi}\xi\cos^2\phi\right]$$



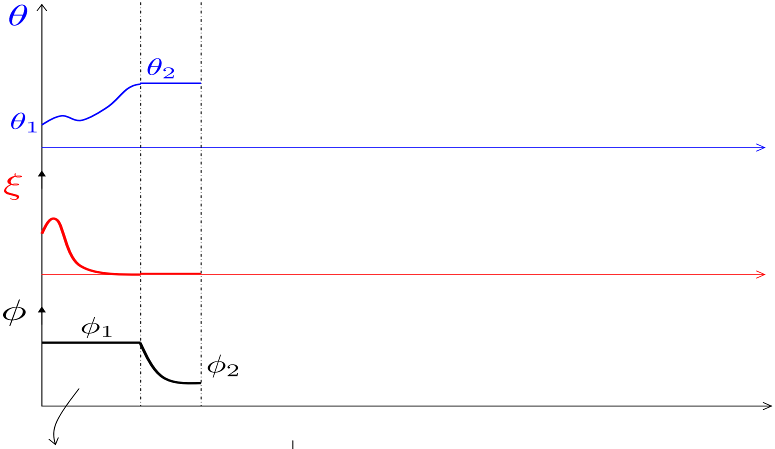
Analysis (3)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix}_{|\xi=0} = 0$$

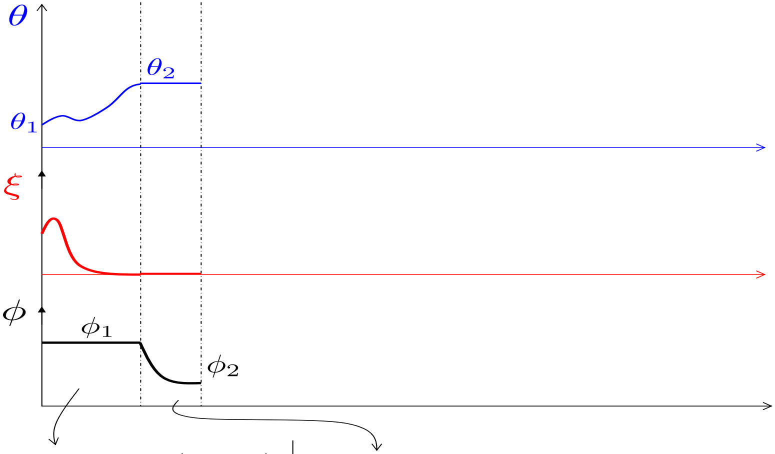
Under zero ξ, the regulated configuration is invariant $\forall\phi(\cdot)$



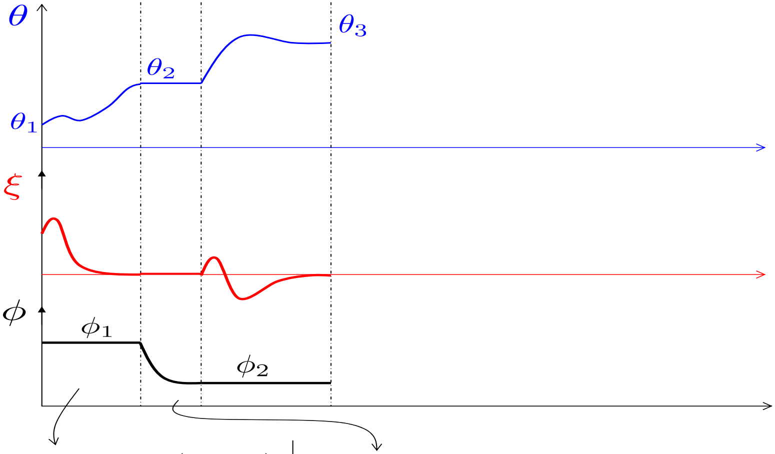
$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\left(F(\theta_1,\theta_2)\right)$$



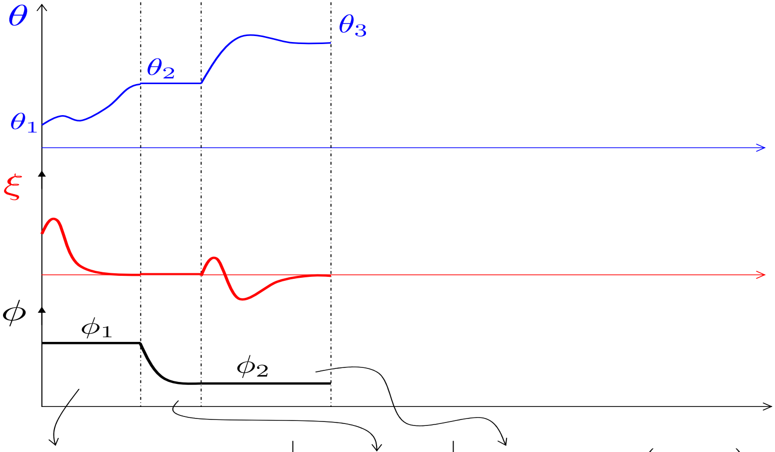
$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\left(F(\theta_1,\theta_2)\right)$$



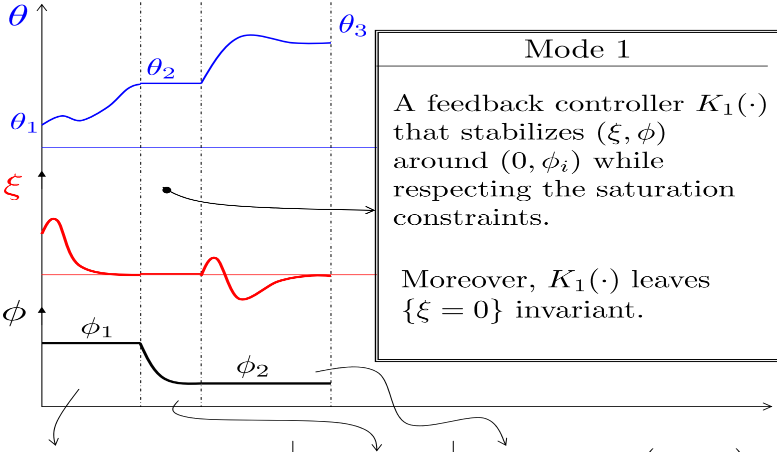
$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\Big(F(\theta_1,\theta_2)\Big) \quad\Bigg|\quad \begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\Big(F(\theta_1,\theta_2)\Big) \quad\Bigg|\quad \begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\Big(F(\theta_1,\theta_2)\Big) \quad\Bigg|\quad \begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad\Bigg|\quad \begin{pmatrix}\Delta_2\bar{x}\\ \Delta_2\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_2}\Big(F(\theta_2,\theta_3)\Big)$$

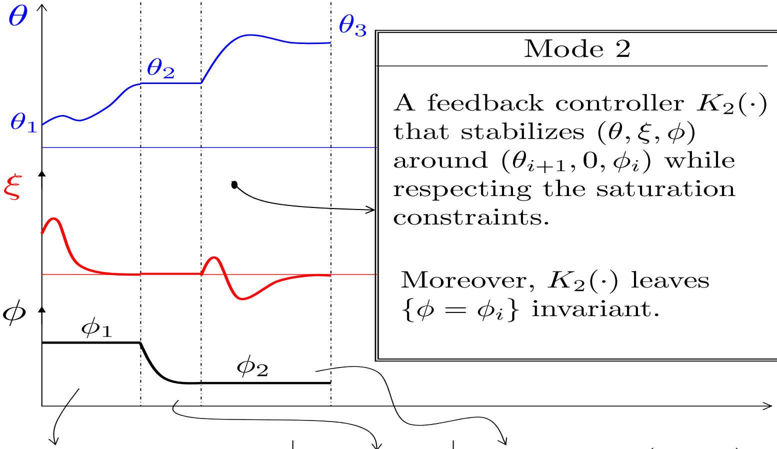


Mode 1

A feedback controller $K_1(\cdot)$ that stabilizes (ξ, ϕ) around $(0, \phi_i)$ while respecting the saturation constraints.

Moreover, $K_1(\cdot)$ leaves $\{\xi = 0\}$ invariant.

$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\Big(F(\theta_1,\theta_2)\Big) \quad \Bigg| \quad \begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \Bigg| \quad \begin{pmatrix}\Delta_2\bar{x}\\ \Delta_2\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_2}\Big(F(\theta_2,\theta_3)\Big)$$

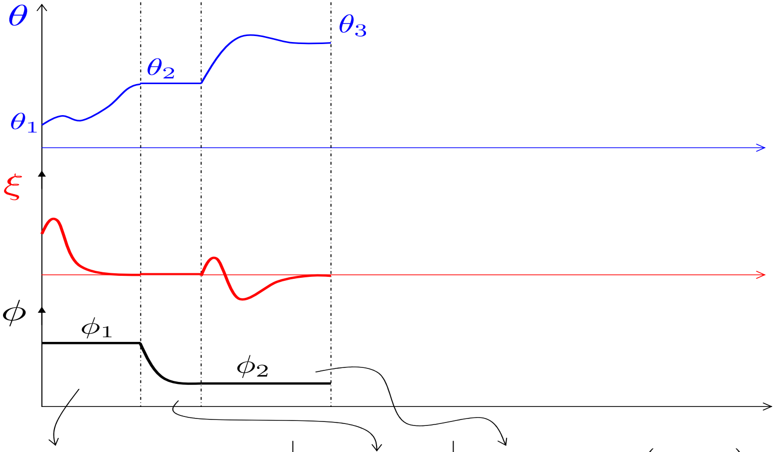


Mode 2

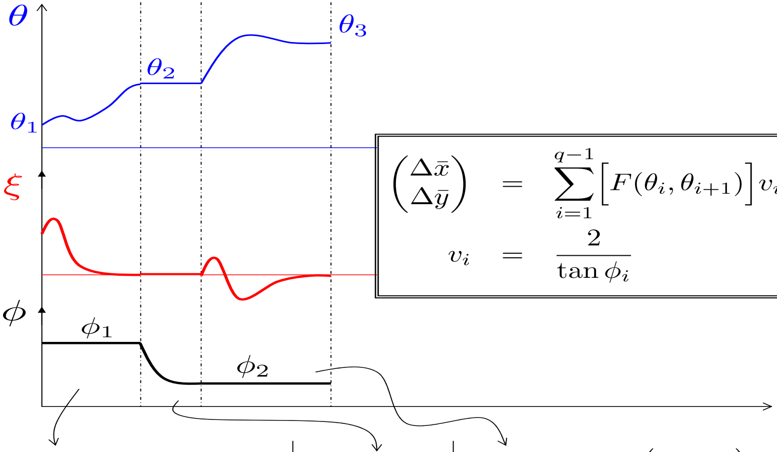
A feedback controller $K_2(\cdot)$ that stabilizes (θ, ξ, ϕ) around $(\theta_{i+1}, 0, \phi_i)$ while respecting the saturation constraints.

Moreover, $K_2(\cdot)$ leaves $\{\phi = \phi_i\}$ invariant.

$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\begin{pmatrix}F(\theta_1,\theta_2)\end{pmatrix} \quad\Bigg|\quad \begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad\Bigg|\quad \begin{pmatrix}\Delta_2\bar{x}\\ \Delta_2\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_2}\begin{pmatrix}F(\theta_2,\theta_3)\end{pmatrix}$$



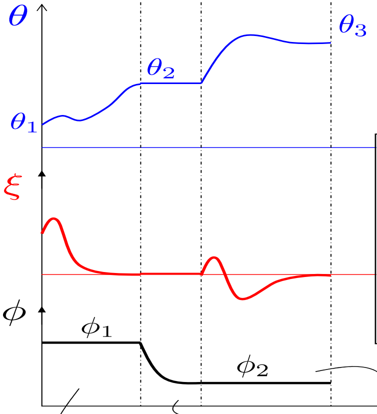
$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\Big(F(\theta_1,\theta_2)\Big) \quad \Bigg| \quad \begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \Bigg| \quad \begin{pmatrix}\Delta_2\bar{x}\\ \Delta_2\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_2}\Big(F(\theta_2,\theta_3)\Big)$$



$$\begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \sum_{i=1}^{q-1}\Big[F(\theta_i,\theta_{i+1})\Big]v_i$$

$$v_i = \frac{2}{\tan\phi_i}$$

$$\begin{pmatrix}\Delta_1\bar{x}\\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\Big(F(\theta_1,\theta_2)\Big) \quad\Bigg|\quad \begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad\Bigg|\quad \begin{pmatrix}\Delta_2\bar{x}\\ \Delta_2\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_2}\Big(F(\theta_2,\theta_3)\Big)$$



Given a sequence

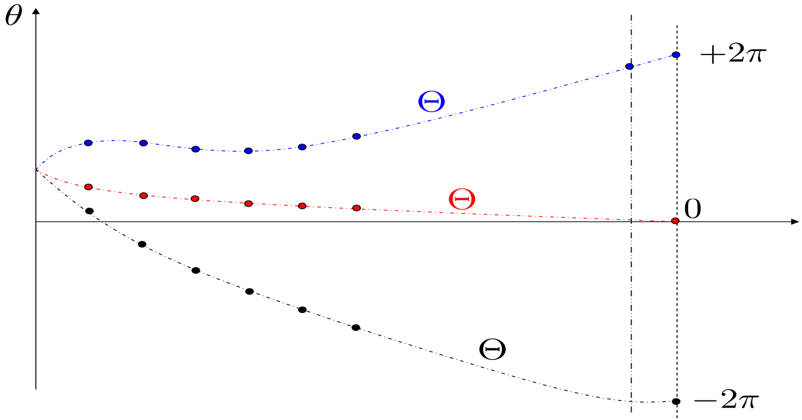
$$\Theta := \left\{\theta_i\right\}_{i=1}^{q} \qquad \begin{array}{rcl}\theta_1 & = & \theta_0 \\ \theta_q & = & 0 \ (mod\ 2\pi)\end{array}$$

$$\begin{pmatrix}\Delta\bar{x} \\ \Delta\bar{y}\end{pmatrix} = \left[M(\Theta)\right] \begin{pmatrix} v_1 \\ \vdots \\ v_{q-1}\end{pmatrix}$$

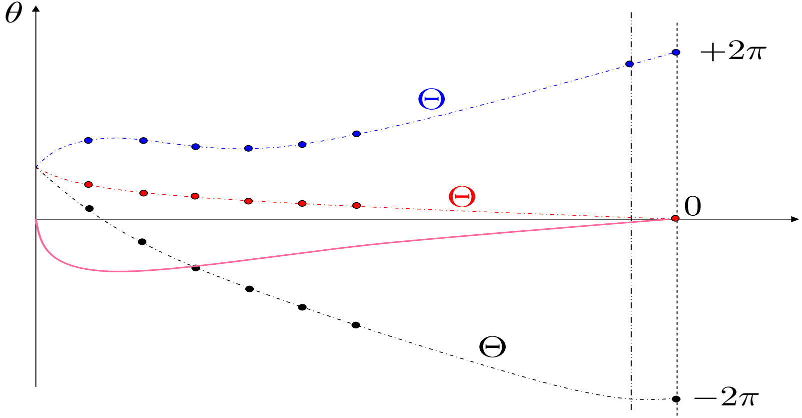
$$v_i = \frac{2}{\tan\phi_i}$$

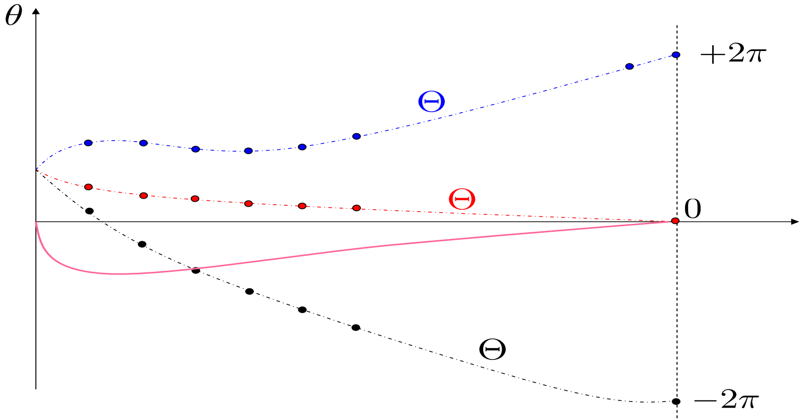
$M(\Theta)$ Non singular

$$\begin{pmatrix}\Delta_1\bar{x} \\ \Delta_1\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_1}\Big(F(\theta_1,\theta_2)\Big) \quad \Bigg| \quad \begin{pmatrix}\Delta\bar{x} \\ \Delta\bar{y}\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \quad \Bigg| \quad \begin{pmatrix}\Delta_2\bar{x} \\ \Delta_2\bar{y}\end{pmatrix} = \frac{2}{\tan\phi_2}\Big(F(\theta_2,\theta_3)\Big)$$

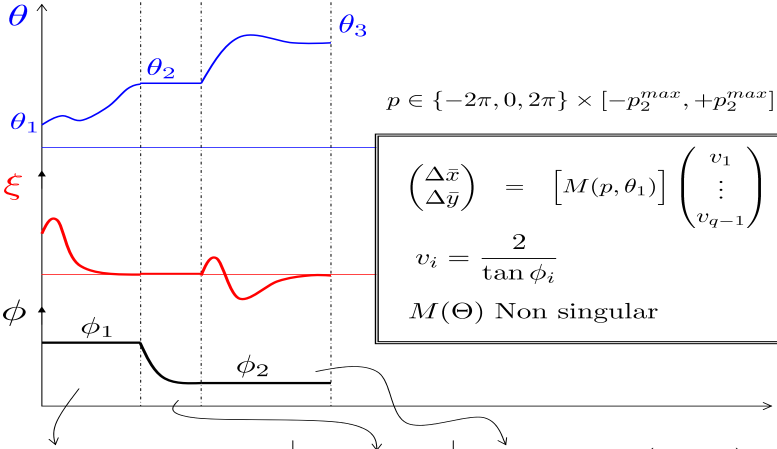


$$\Theta(t, p, \theta^0) := 2p_1\pi + (\theta^0 - 2p_1\pi)e^{-\lambda_\theta t}$$


θ
$+2\pi$
Θ
Θ
0
Θ
-2π



$$\Theta(t,p,\theta^0) := 2p_1\pi + (\theta^0 - 2p_1\pi)e^{-\lambda_\theta t} + p_2 e^{-\lambda_\theta t}\left(1 - e^{-\lambda_\theta t}\right)$$



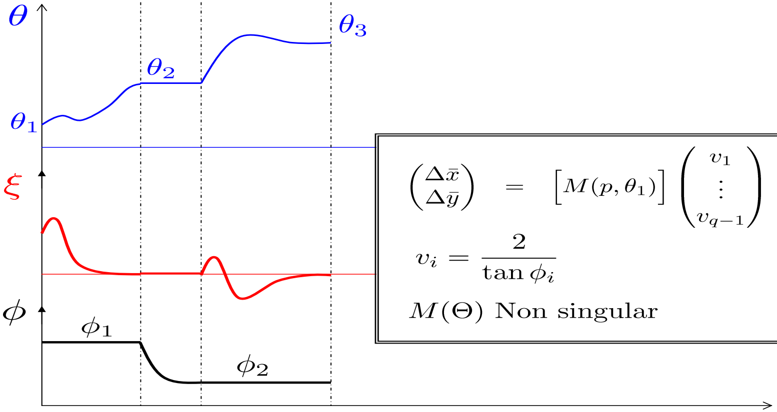
$$p \in \{-2\pi, 0, 2\pi\} \times [-p_2^{max}, +p_2^{max}]$$

$$\begin{pmatrix} \Delta\bar{x} \\ \Delta\bar{y} \end{pmatrix} = \Big[M(p,\theta_1) \Big] \begin{pmatrix} v_1 \\ \vdots \\ v_{q-1} \end{pmatrix}$$

$$v_i = \frac{2}{\tan\phi_i}$$

$M(\Theta)$ Non singular

$$\begin{pmatrix} \Delta_1\bar{x} \\ \Delta_1\bar{y} \end{pmatrix} = \frac{2}{\tan\phi_1} \Big(F(\theta_1,\theta_2) \Big) \quad \Bigg| \quad \begin{pmatrix} \Delta\bar{x} \\ \Delta\bar{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} \Delta_2\bar{x} \\ \Delta_2\bar{y} \end{pmatrix} = \frac{2}{\tan\phi_2} \Big(F(\theta_2,\theta_3) \Big)$$



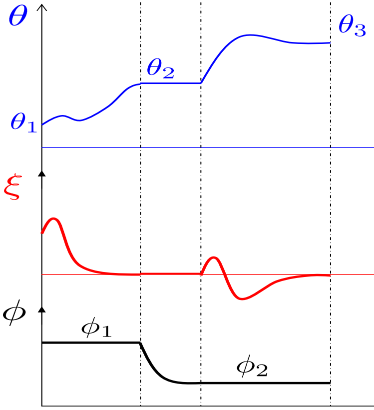
$$\begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \left[M(p,\theta_1)\right]\begin{pmatrix}v_1\\ \vdots \\ v_{q-1}\end{pmatrix}$$

$$v_i = \frac{2}{\tan\phi_i}$$

$M(\Theta)$ Non singular

$$\hat{v}(p,x^0) := \arg\min_{\tilde{v}} \left[-\sum_{i=1}^{q}\tilde{v}_i\right] \quad \text{under} \quad \left|\begin{array}{l}\begin{pmatrix}\bar{x}^0\\ \bar{y}^0\end{pmatrix} + M(p,\theta^0)\tilde{v} = 0\\ \|\tilde{v}\|_\infty \le v_{max}\end{array}\right.$$

LP Problem



$$\hat{p}(x^0) := arg\min_p J\Big(x^0, \hat{v}(p, x^0)\Big)$$

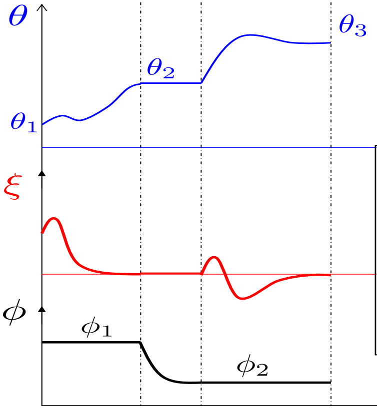
Weighted excursion in the $(\bar{x}, \bar{y}, \theta)$ plane

$$\begin{pmatrix}\Delta\bar{x}\\ \Delta\bar{y}\end{pmatrix} = \Big[M(p, \theta_1)\Big] \begin{pmatrix}v_1\\ \vdots \\ v_{q-1}\end{pmatrix}$$

$$v_i = \frac{2}{\tan\phi_i}$$

$M(\Theta)$ Non singular

$$\hat{v}(p, x^0) := \arg\min_{\tilde{v}} \Big[-\sum_{i=1}^{q} \tilde{v}_i\Big] \quad \text{under} \quad \left| \begin{array}{l} \begin{pmatrix}\bar{x}^0\\ \bar{y}^0\end{pmatrix} + M(p, \theta^0)\tilde{v} = 0 \\ \|\tilde{v}\|_\infty \leq v_{max} \end{array} \right.$$



$$\hat{p}(x^0) := arg \min_p J\Big(x^0, \hat{v}(p, x^0)\Big)$$

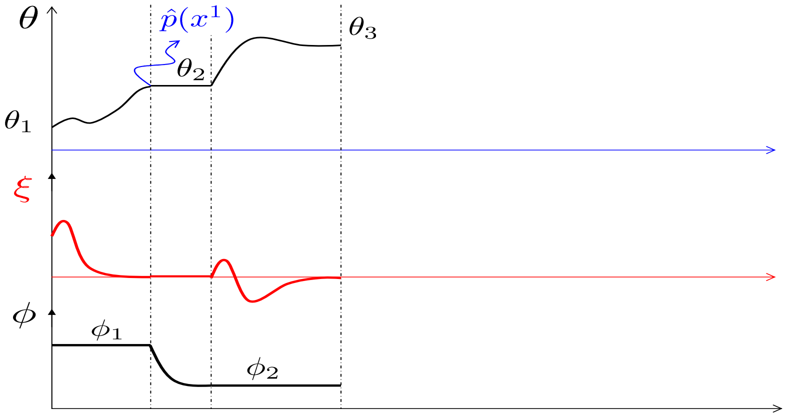
Weighted excursion in the $(\bar{x}, \bar{y}, \theta)$ plane

$p \in \{-2\pi, 0, 2\pi\} \times [-p_2^{max}, p_2^{max}]$

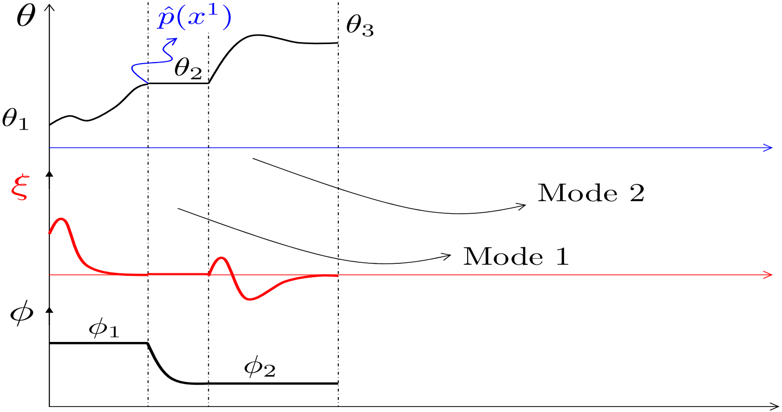
Generation of Open-loop *"steering trajectories"*

3 scalar optimization problems. The cost function is obtained by solving an LP problem.

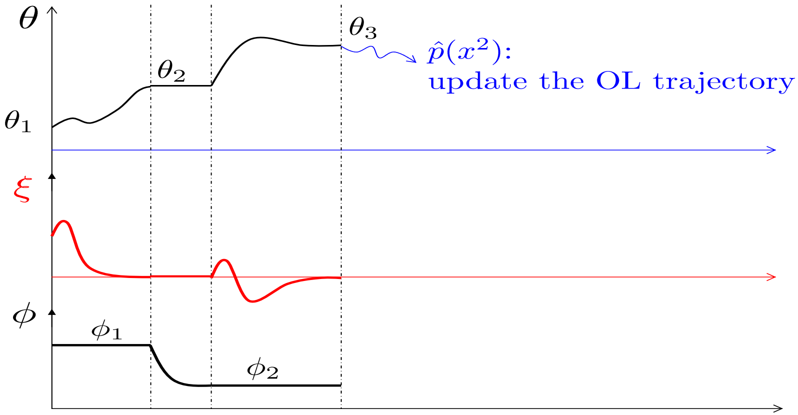
$$\hat{v}(p, x^0) := \arg\min_{\tilde{v}} \Big[-\sum_{i=1}^{q} \tilde{v}_i\Big] \quad \text{under} \quad \left| \begin{array}{l} \begin{pmatrix} \bar{x}^0 \\ \bar{y}^0 \end{pmatrix} + M(p, \theta^0)\tilde{v} = 0 \\ \|\tilde{v}\|_\infty \leq v_{max} \end{array} \right.$$



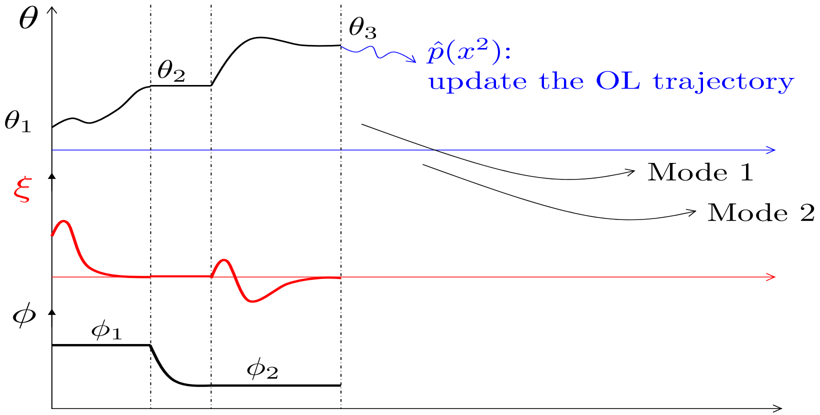
Receding Horizon Implementation



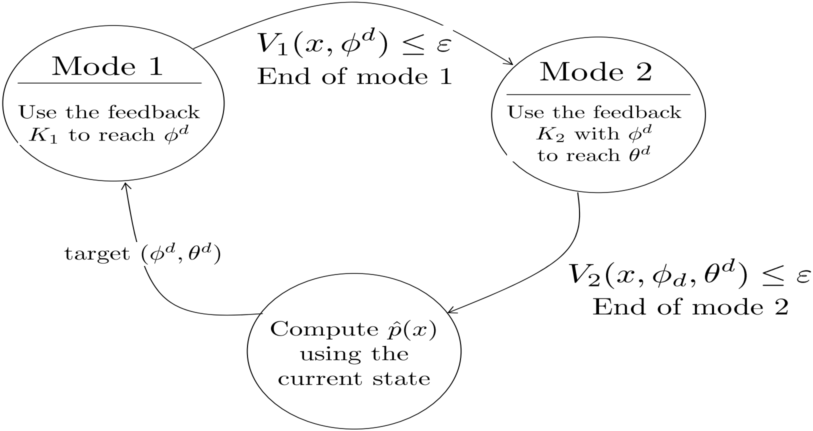
Receding Horizon Implementation



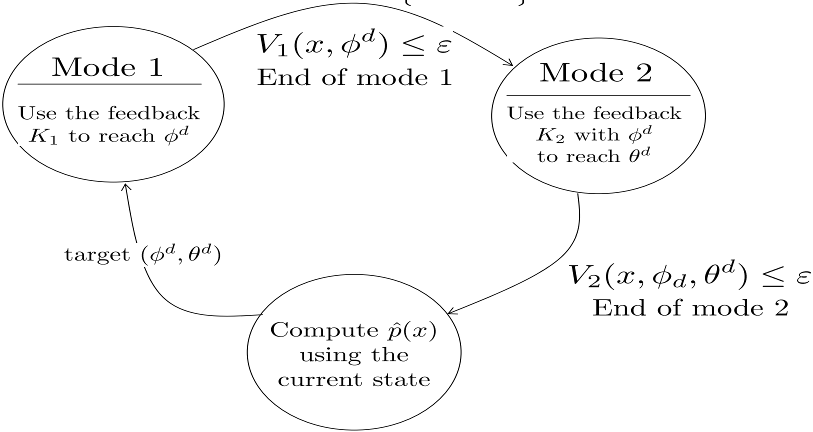
Receding Horizon Implementation



Receding Horizon Implementation

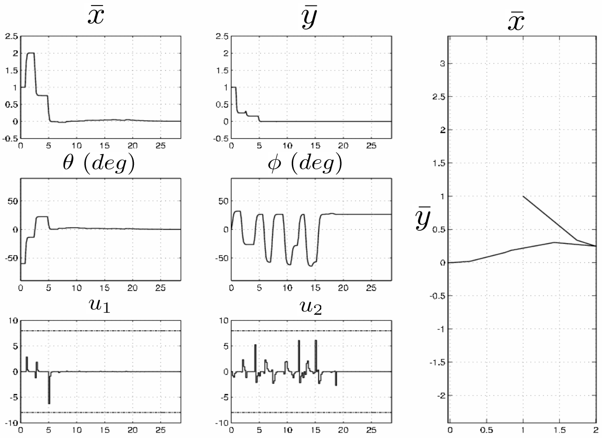

Mode 1
Use the feedback K_1 to reach ϕ^d
$V_1(x, \phi^d) \leq \varepsilon$
End of mode 1
Mode 2
Use the feedback K_2 with ϕ^d to reach θ^d
$V_2(x, \phi_d, \theta^d) \leq \varepsilon$
End of mode 2
Compute $\hat{p}(x)$ using the current state
target (ϕ^d, θ^d)

$$V_1(x,\phi^d) = \max\Big\{|\phi-\phi^d|, |\dot{\phi}|, |\xi|\Big\}$$
$$V_2(x,\phi^d,\theta^d) := \max\Big\{|\theta-\theta^d|, |\xi|\Big\}$$



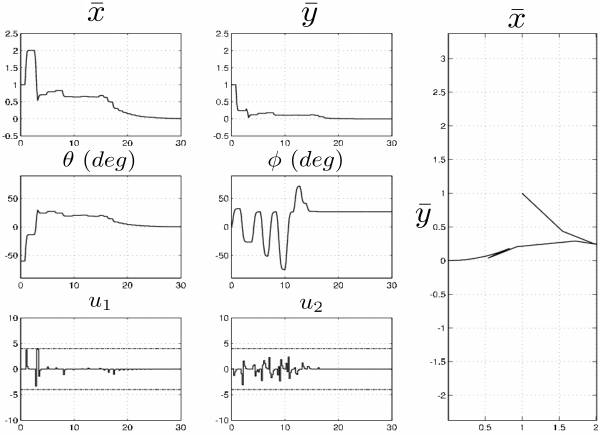
Handling Constraints

Handling Constraints



$$u_1^{max} = u_2^{max} = 8$$

Handling Constraints



$$u_1^{max} = u_2^{max} = 4$$

Performing "*economic trajectories*"

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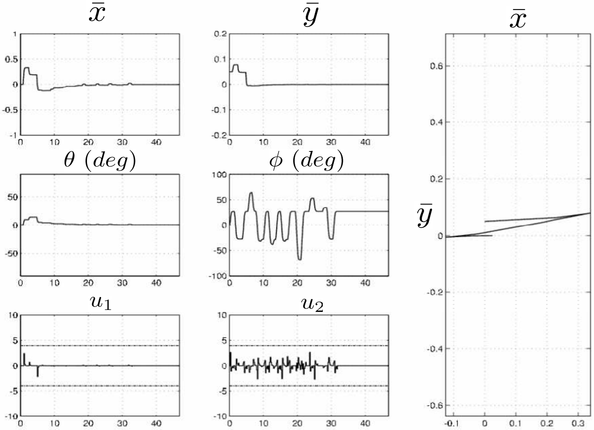
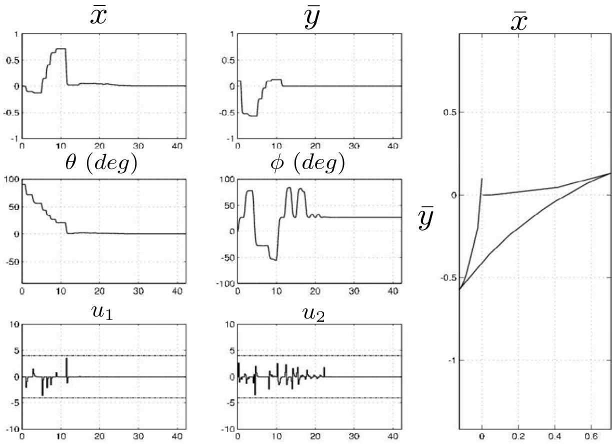


Performing "*economic trajectories*"



Robustness

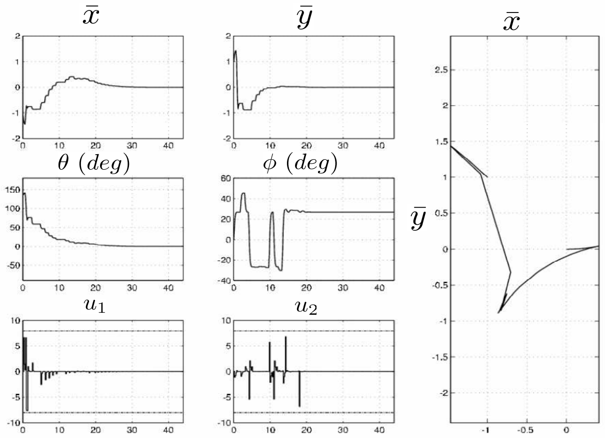
Model uncertainties

$$\dot{\xi} = \left(-\frac{J_r}{2m^2}(1+\delta_1)\ddot{\psi} + \dot{\phi}\xi\right) \cdot \tan\phi + \delta_2$$
$$u_2 = 2J_w(1+\delta_3)\ddot{\phi}$$

where

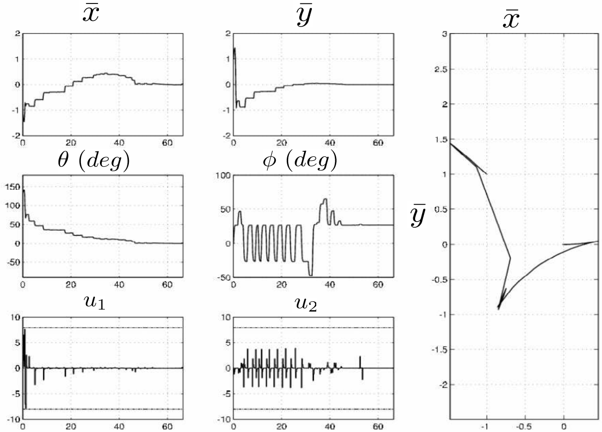
$$\delta_1 = -0.1 \quad ; \quad \delta_2 = 0.05 \quad ; \quad \delta_3 = -0.1$$

Robustness



$\delta_1 = 0$; $\delta_2 = 0$; $\delta_3 = 0$

Robustness



$\delta_1 = -0.1$; $\delta_2 = 0.05$; $\delta_3 = -0.1$

In the paper

- ✓ More general setting
- ✓ Rigorous set of assumptions
- ✓ Expressions of the constrained feedbacks K_1 and K_2
- ✓ Sketch of the convergence proof (practical stability)
- ✓ Justification of some choices

Future work

- Moving Obstacle Avoidance
- Micro-Prototype ?
  "Flying" and 2D movement using the same motor.