Nonlinear Model Predictive Control for Fast Systems

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CNRS / Gipsa-lab / Département d'Automatique

Equipe Systèmes Non Linéaires et Complexité (SYSCO)

JNRR'07

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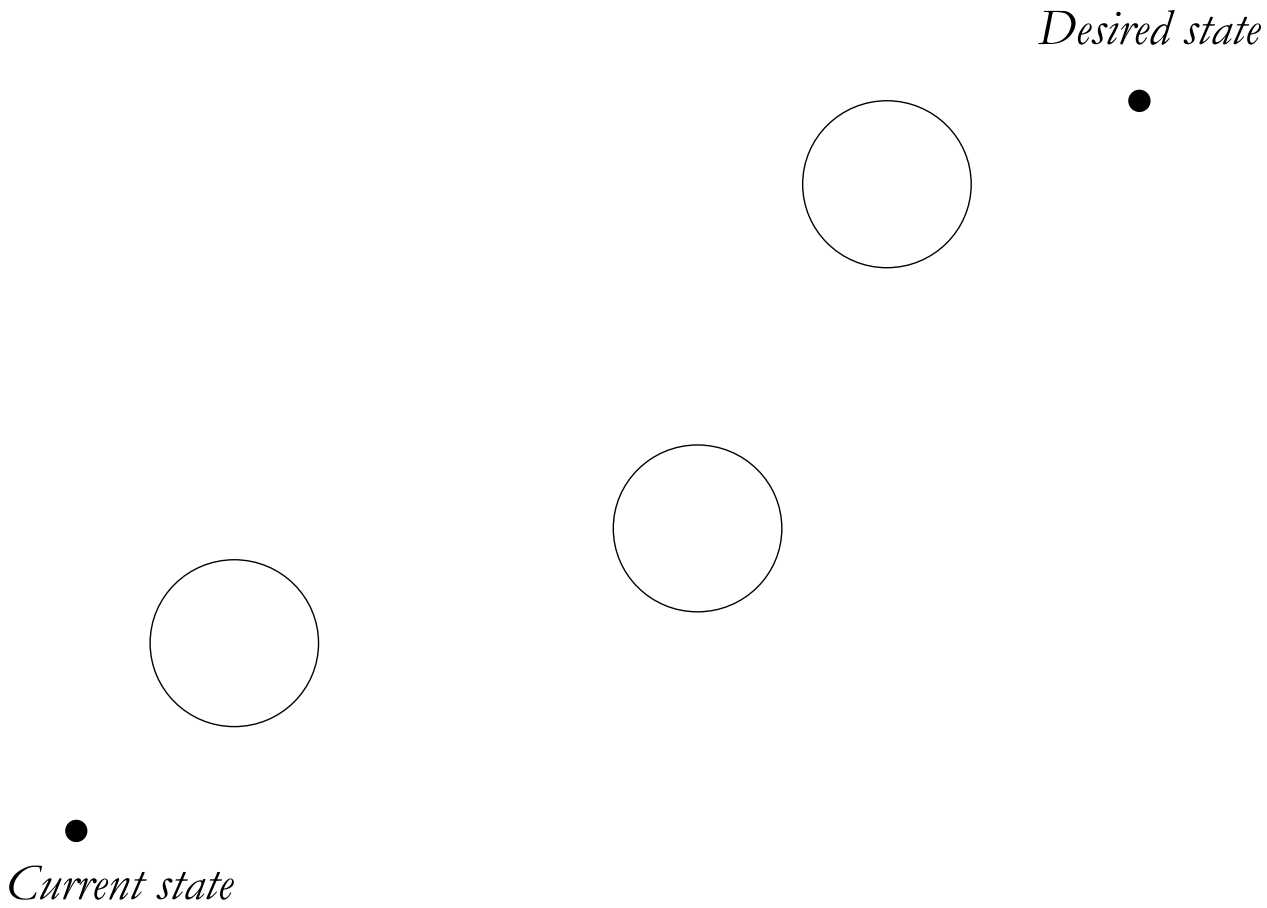

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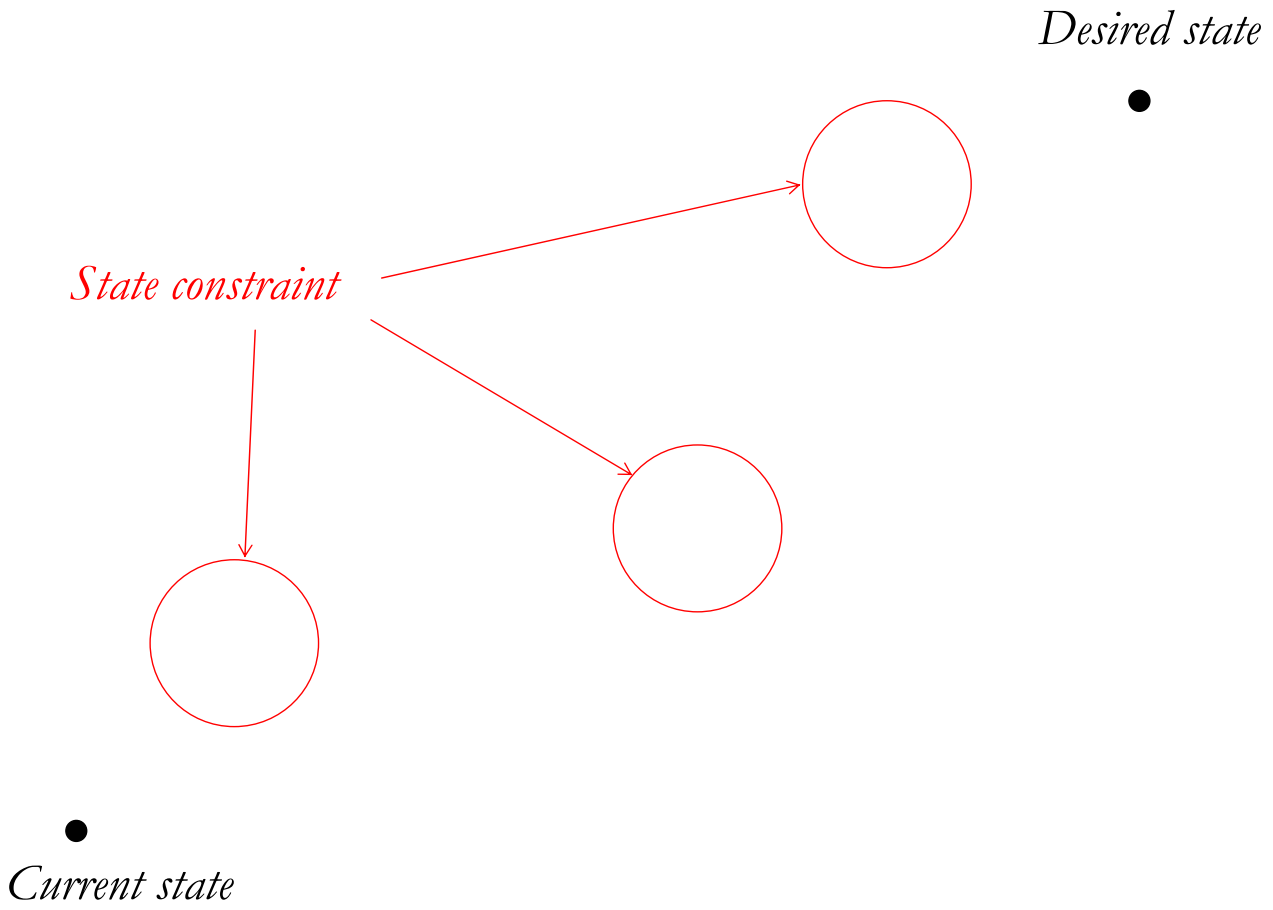

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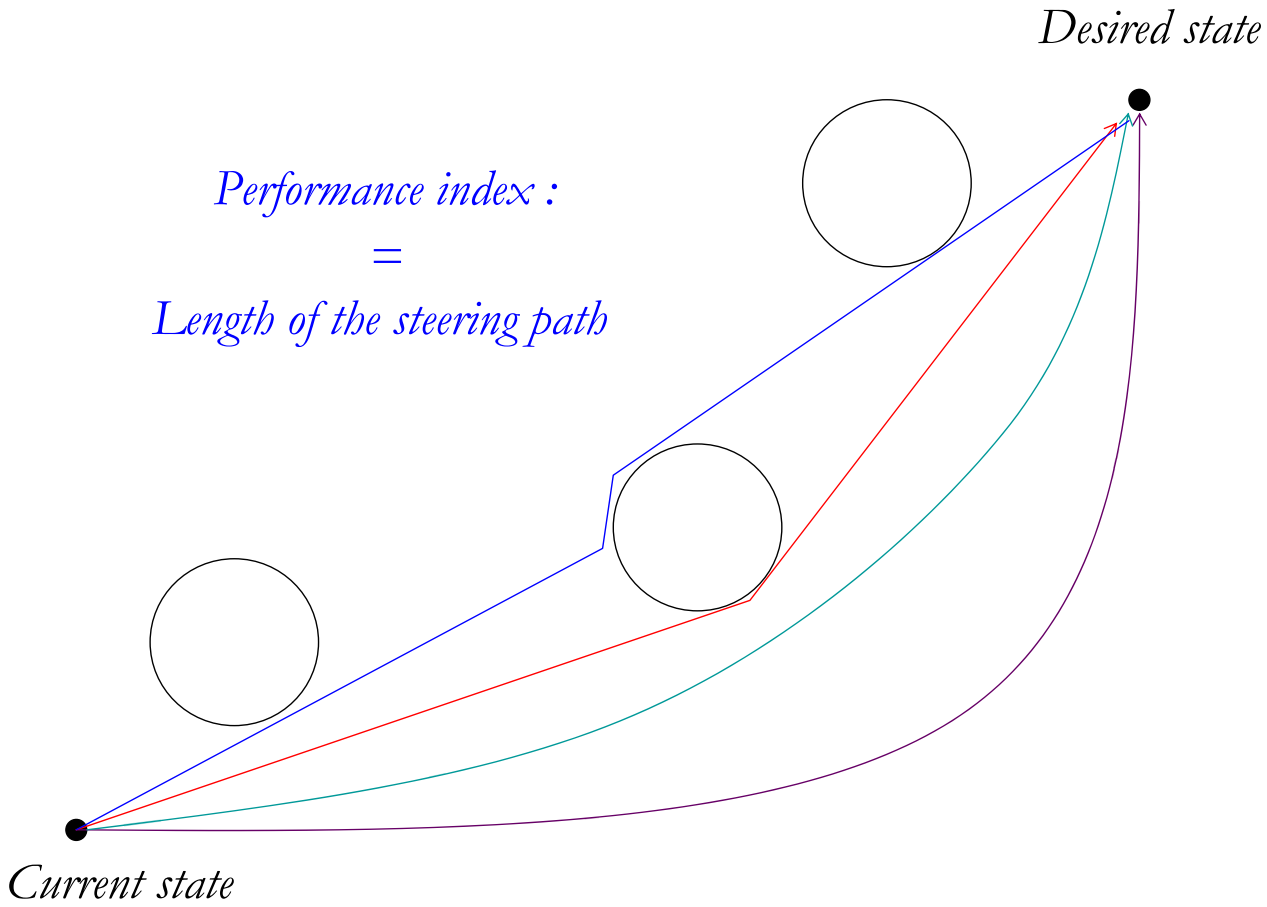

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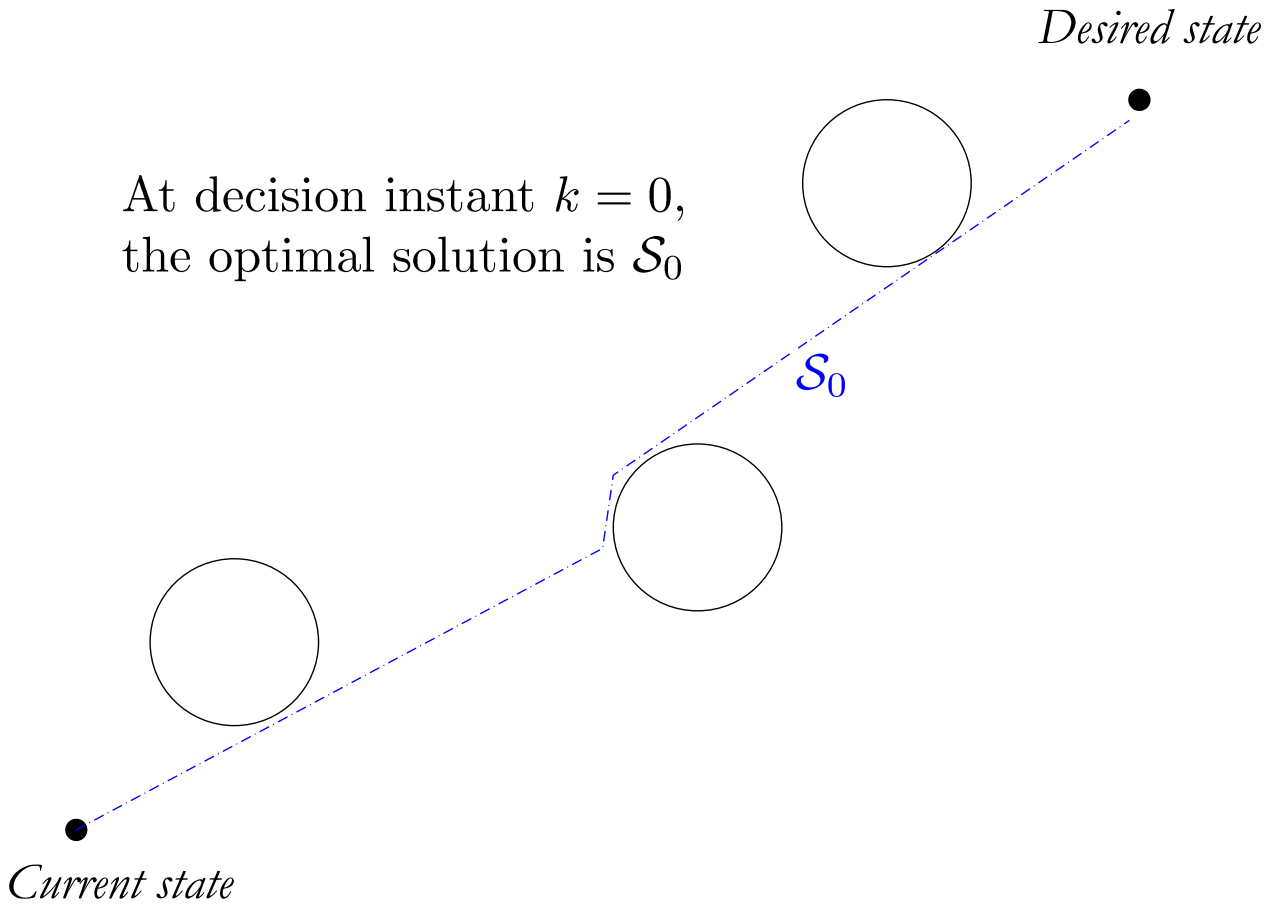

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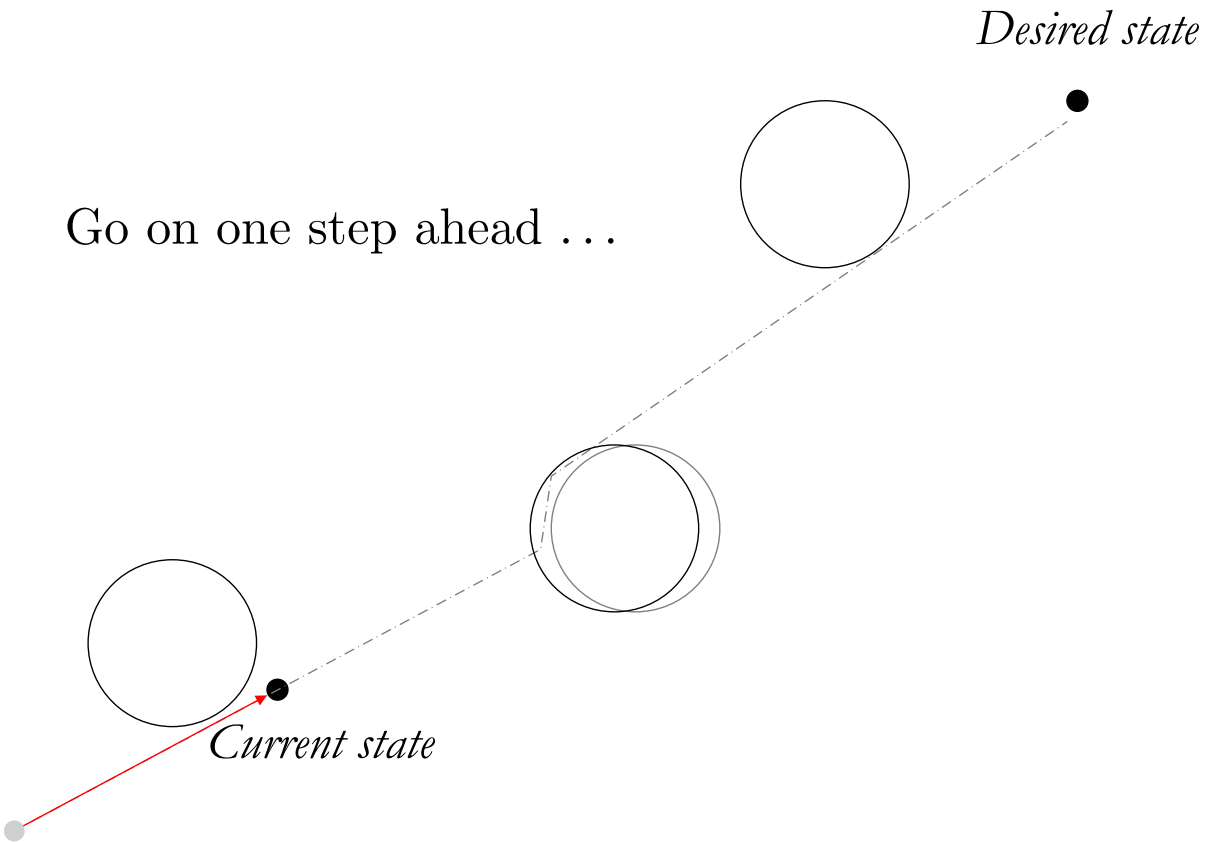

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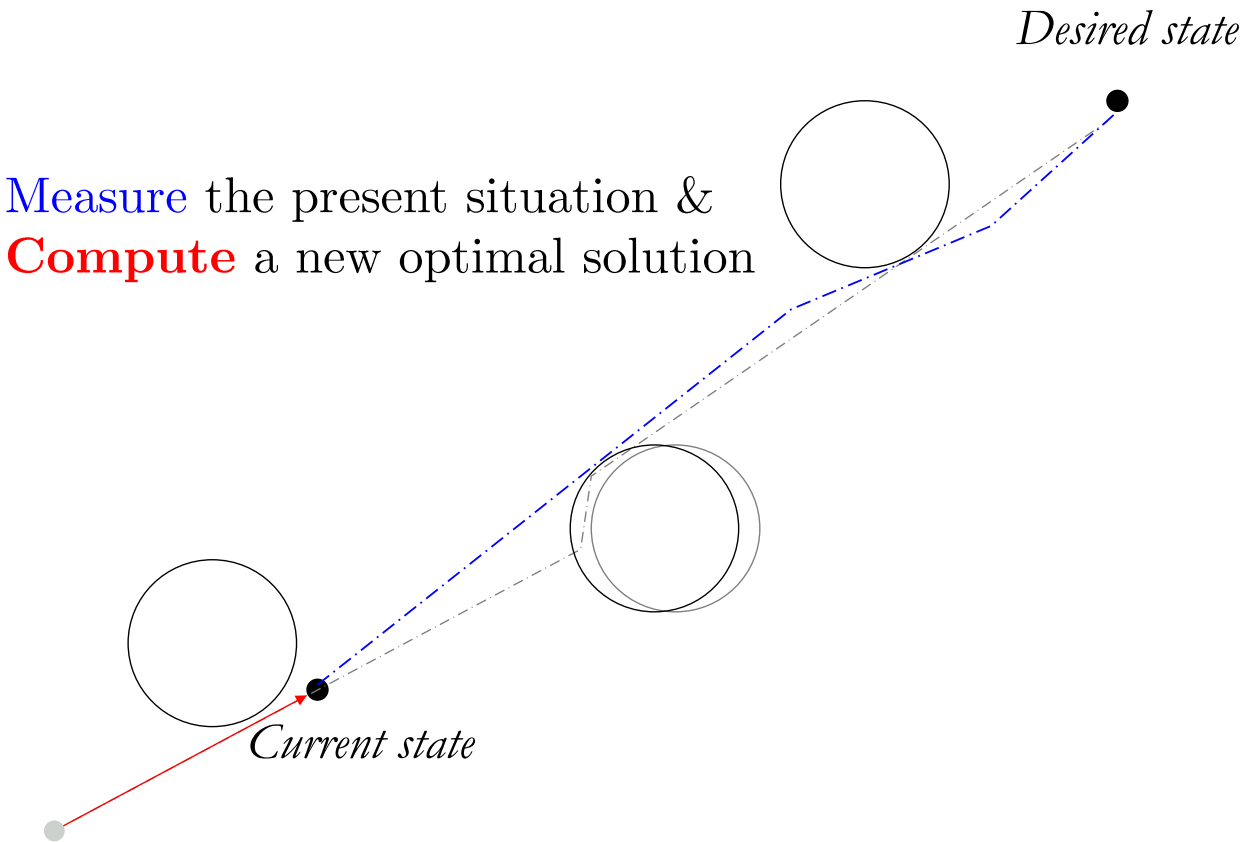

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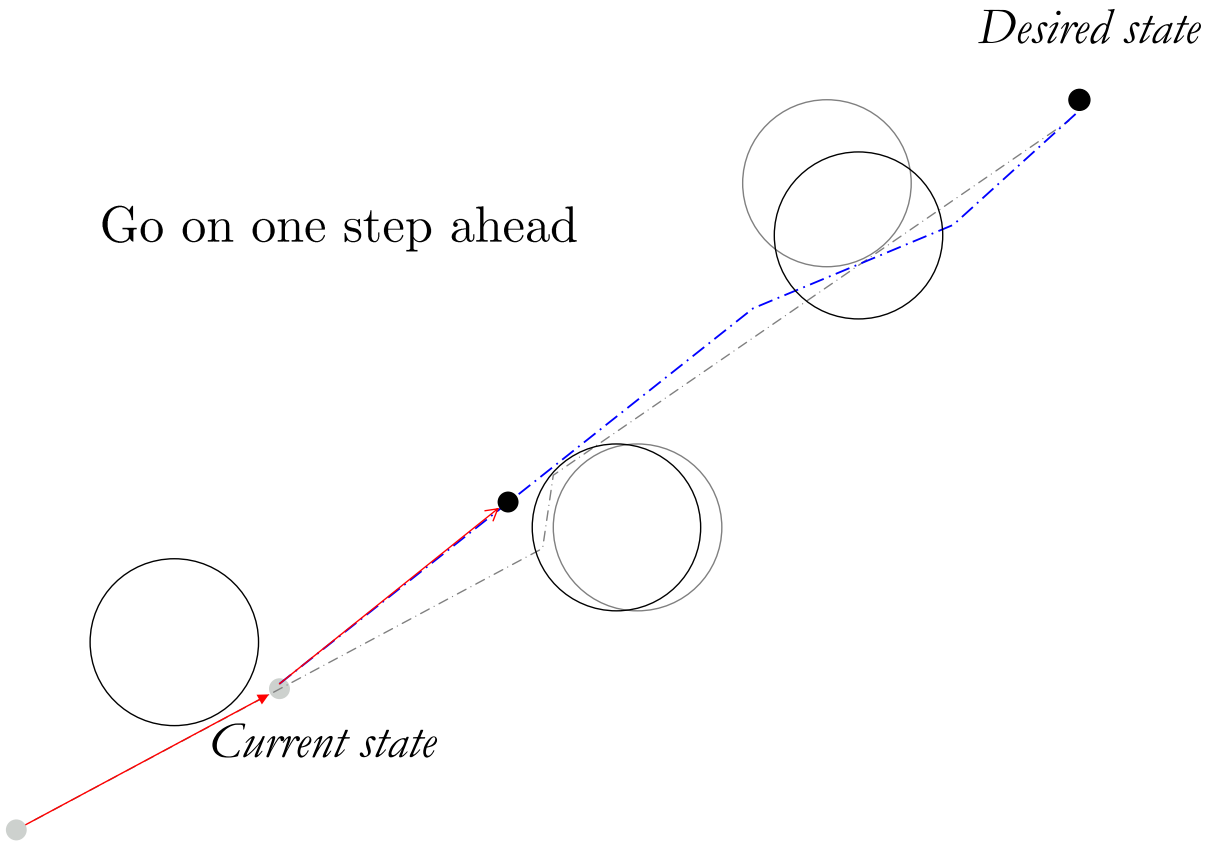

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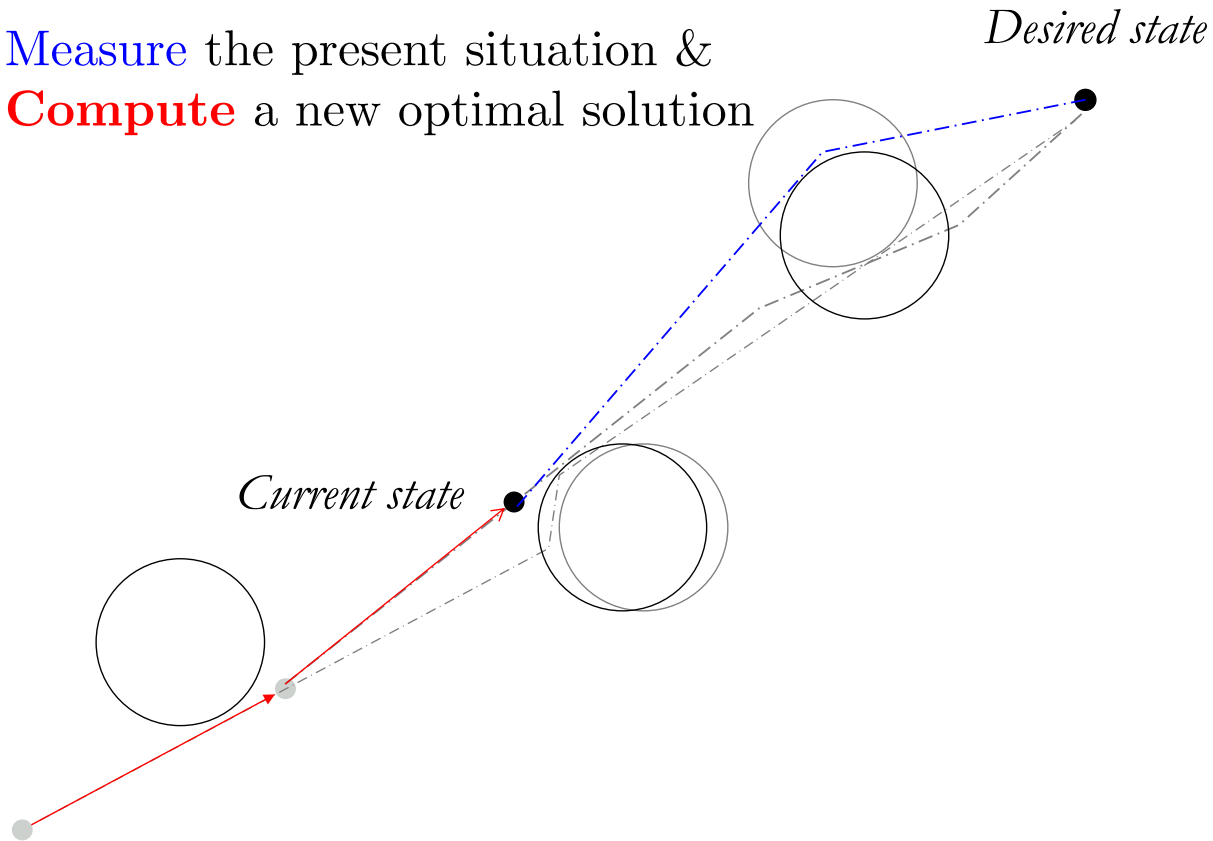

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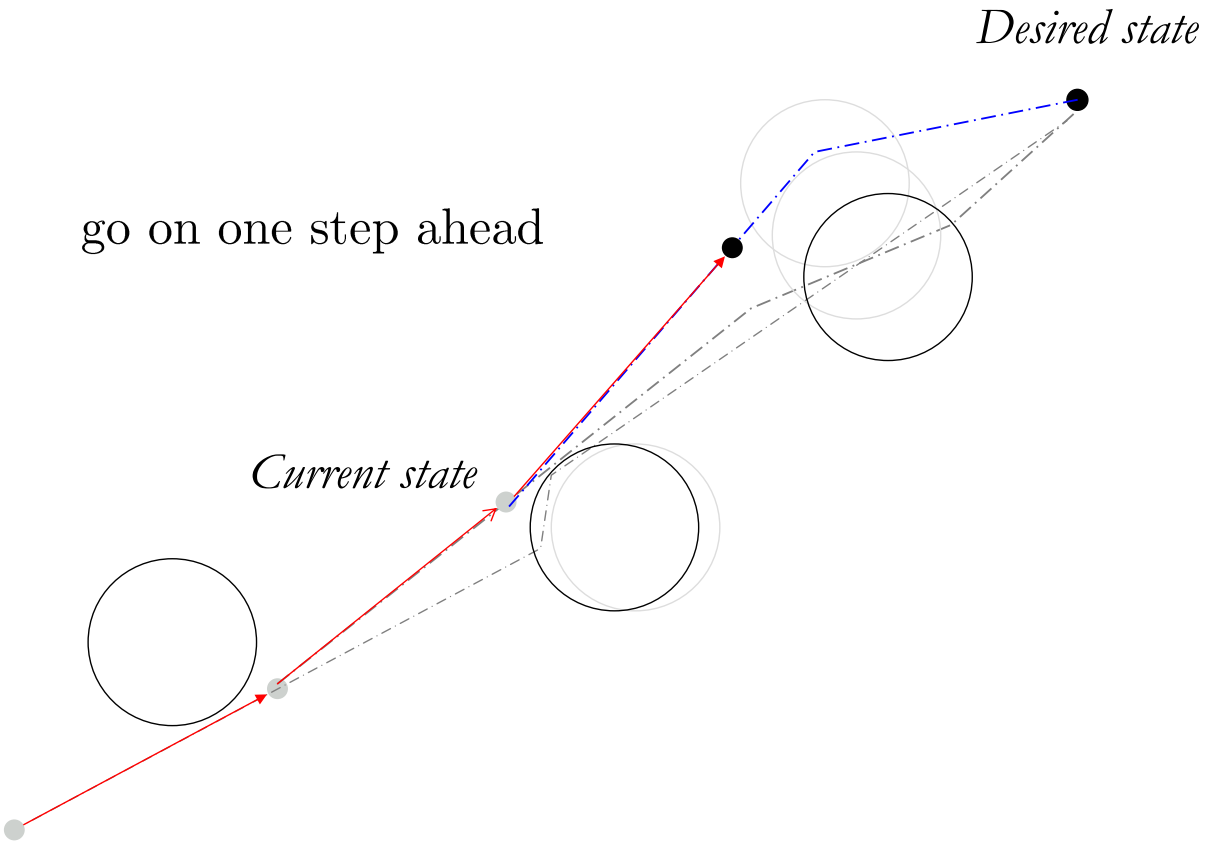

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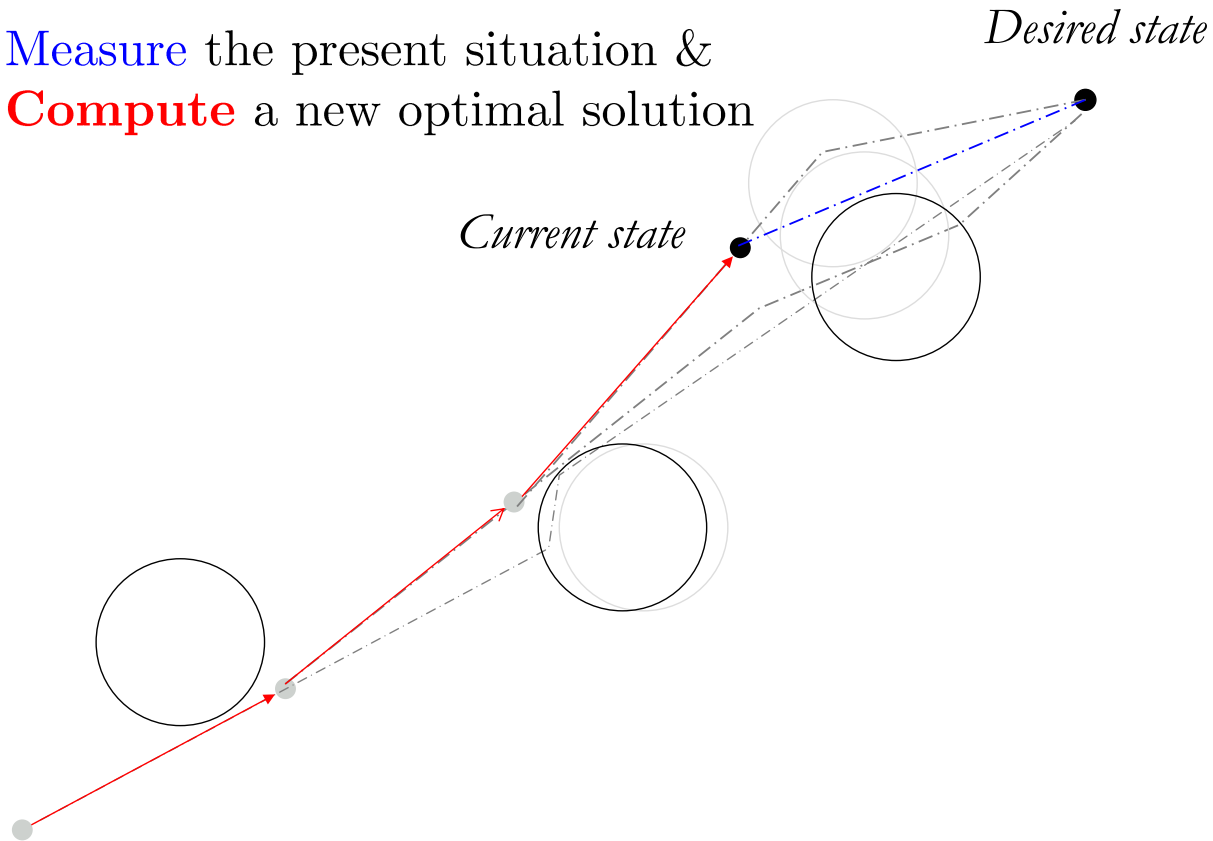

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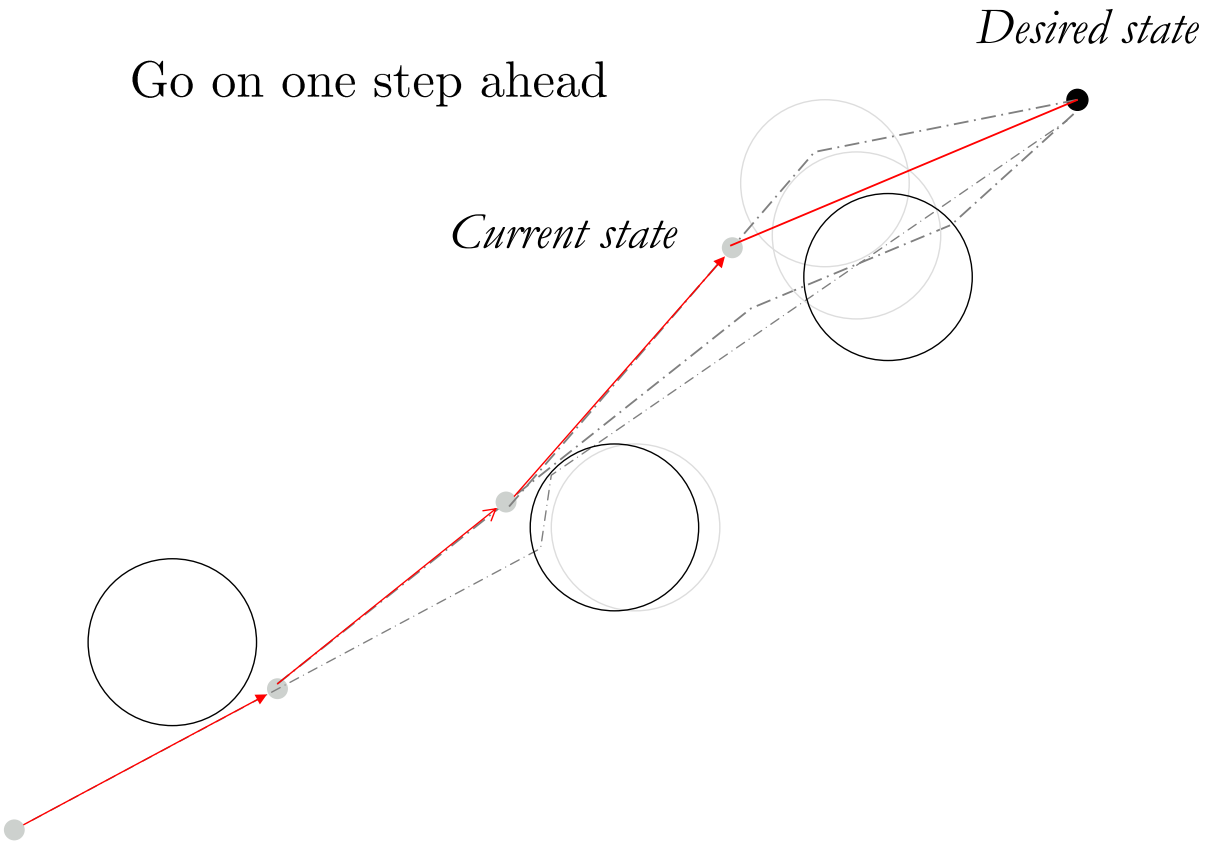

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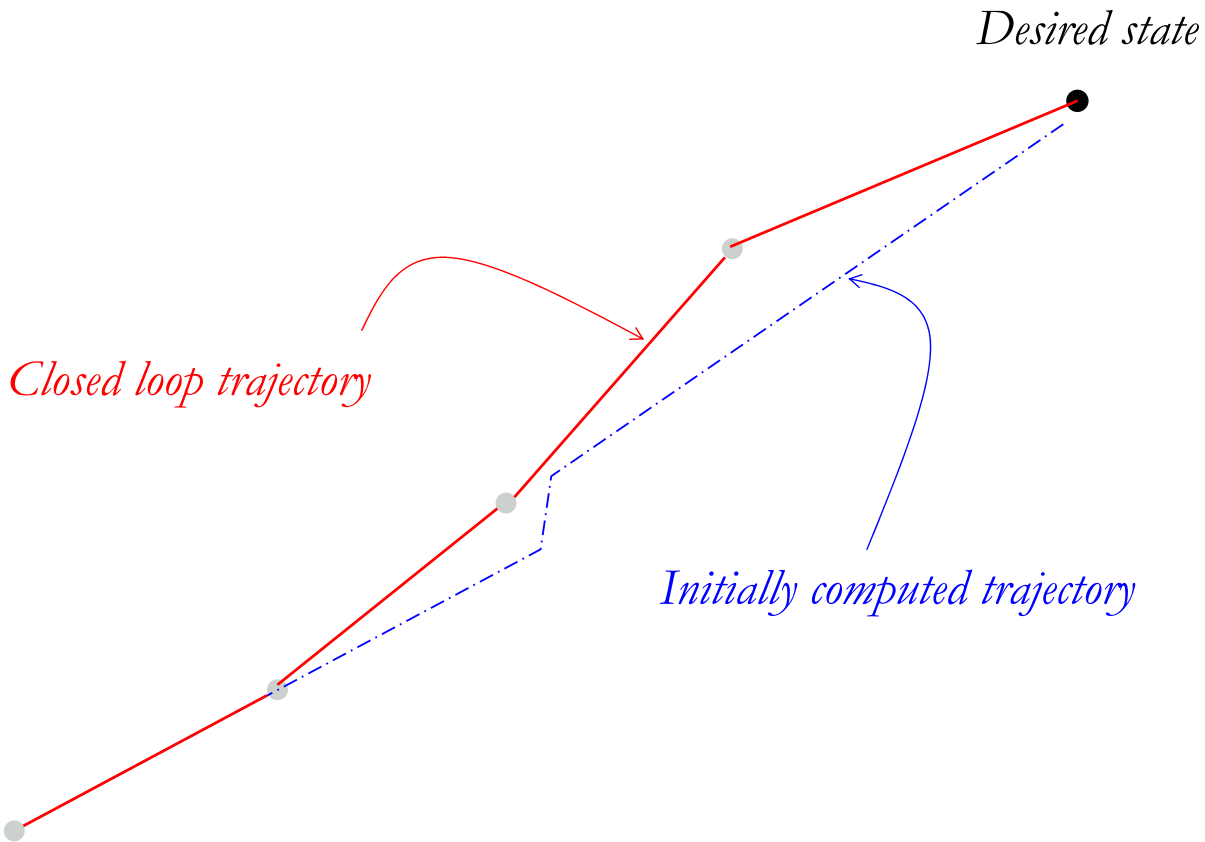

More formally

Consider a dynamical system :

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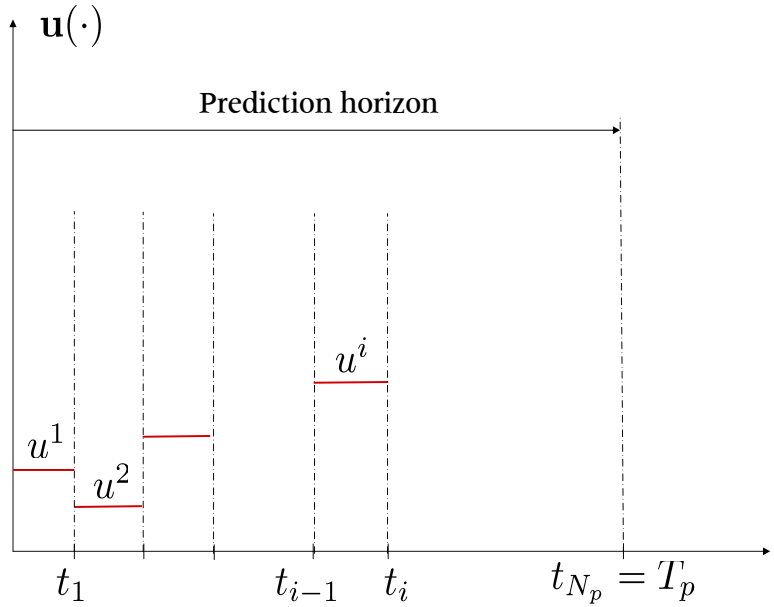

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More generally

Any map

$$C : \mathbb{P} \longrightarrow \mathbb{U}^N$$
$$C(p) = \begin{pmatrix} u^1(p) & \dots & u^N(p) \end{pmatrix}$$

defines a $\mathbb{P}$-parametrized piecewise constant control profile

$$\mathbf{u} = \mathcal{U}_{pwc}(\cdot, p)$$

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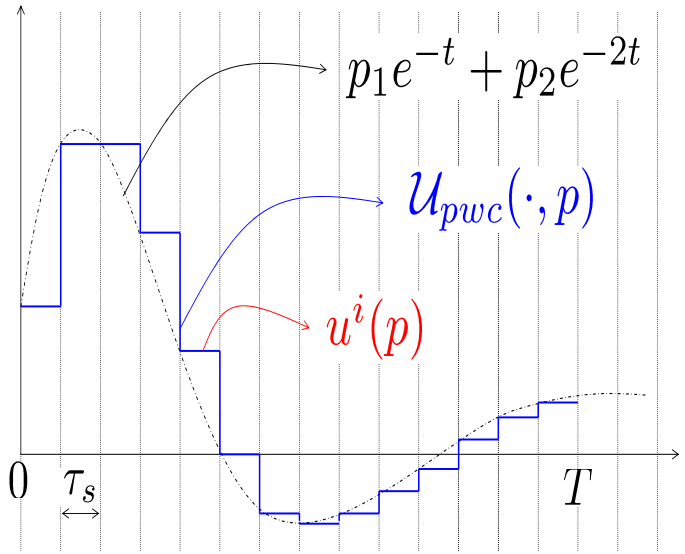

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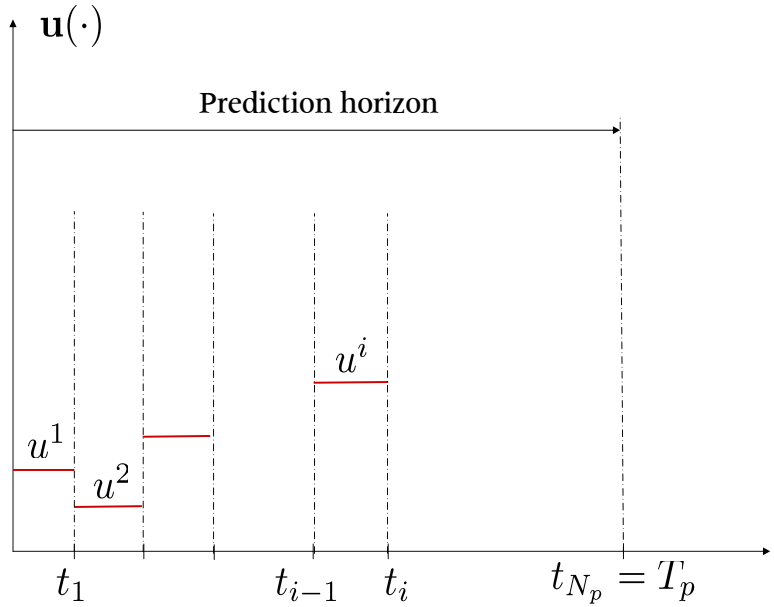

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Advantages

1. No structural assumption on the model
2. Expressing optimality and trade-off
3. Systematic constraint handling

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Price to pay

Solving ON-LINE non convex optimization problems.

Non-exhaustive MPC Vendor List

- ABB
- ACT
- Adaptics
- Adaptive Resources
- Adersa Home Page
- Aspen Technology
- Aurel Systems Inc.
- Batch CAD
- Bonner and Moore
- Brainwave
- C.F. Picou and Associates
- Chemstations
- Comdale Technologies
- Control Arts Inc.
- Control Consulting Inc.
- Control Dynamics Homepage
- Controlsoft Incorporated
- Cybosoft
- DOT Products
- Trieber Controls
- Yokogawa APC
- US Process Control L.L.C.
- Eldridge Engineering Inc.
- Elsag Bailey
- Envision Systems Inc.
- Gensym
- Enterprise Control Technologies
- Fantoft Process Group
- MATHWORKS
- Honeywell
- Hyprotech
- Inferential Control Company
- IntellOpt
- Knowledgescape
- MDC Technology
- Neuralware
- Nexus Engineering
- Objectspace
- Optimal Control Research
- Pavilion Technologies
- Predictive Control Ltd.
- Process System Consultants
- RSI
- Simulation and Advanced Controls Inc.
- Simtech
- Texas Controls Inc.

Typically linear applications and tools

R. Findeisen: NMPC for Fast Nonlinear Systems (Workshop CDC 2006, San Diego)

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The stability is an issue !

Consider the dynamical system

$$\begin{aligned} x_1^+ &= x_1 + (1 + x_2^2)u \\ x_2^+ &= \frac{3}{2}x_2 - x_1 e^u \end{aligned}$$

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- ✓ Open-loop instable.
- ✓ Set of equilibrium states

$$\mathcal{E}_{st} = \left\{ x^{(\alpha)} := \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} \quad ; \quad \alpha \in \mathbb{R} \right\}$$

Control objective starting at $x^{(0)} = (0, 0)$, stabilize the system around $x^{(1)} = (1, 2)$.

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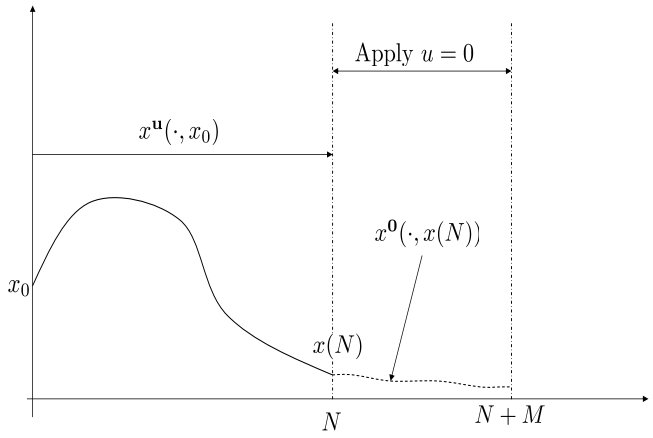
Consider the cost function

$$J(x, \mathbf{u}) := F(x(N)) + \sum_{i=0}^{N} L(x(i), u(i))$$

where

$$L(x, u) := \|x - x^{(1)}\|^2 + ru^2$$

$$F(x) = \sum_{i=1}^{M-N} x^{\mathbf{0}}(i; x)$$



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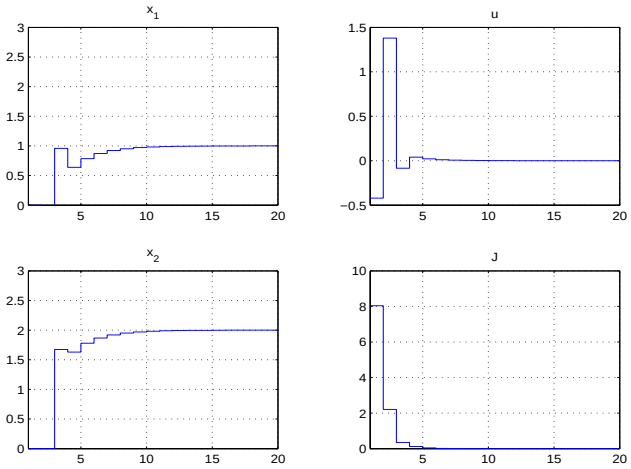
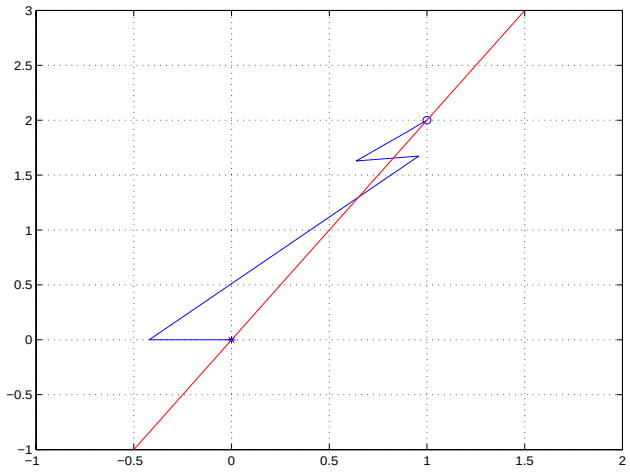
Test 1 : $N = 2$, $M = 2$, $r = 1$





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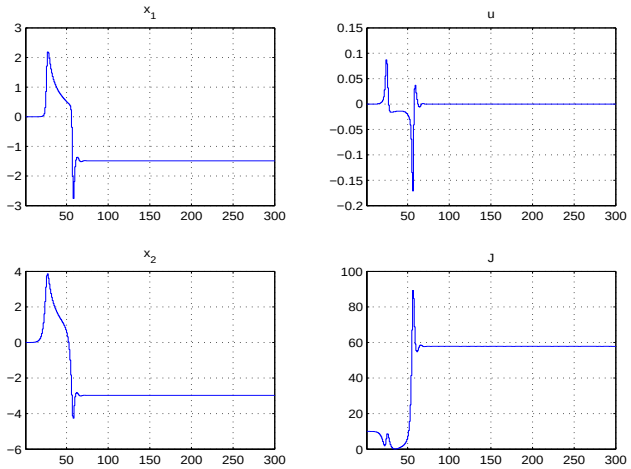
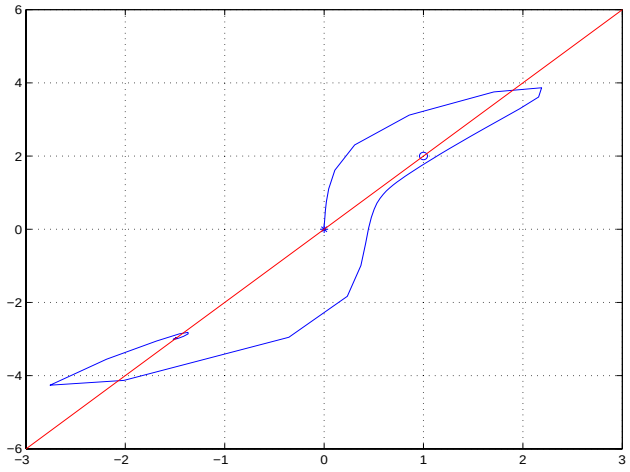
Let us assume that one looks for solution such that $u \leq 0.1$

Assume that for that reason one takes

$$r = 160$$

$N = 2,\ M = 2,\ r = 160$





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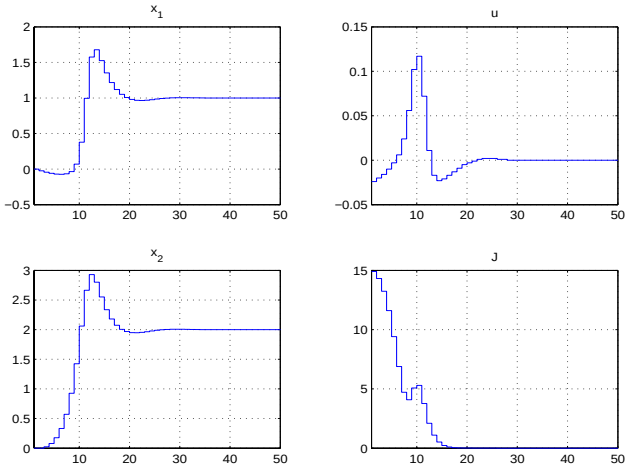
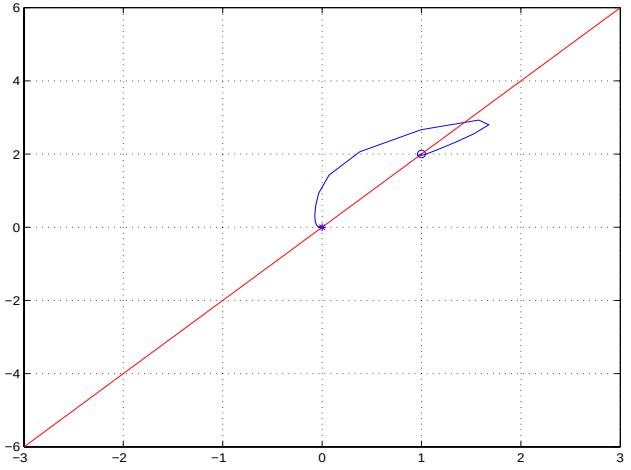
Let us take $M = 3$ (instead of 2)

$N = 2$, $M = 3$, $r = 160$





The stability is an issue !

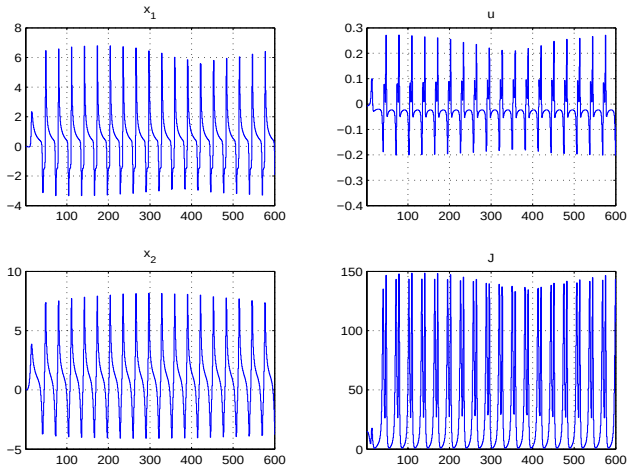
Increasing M enabled the target state to be reached.

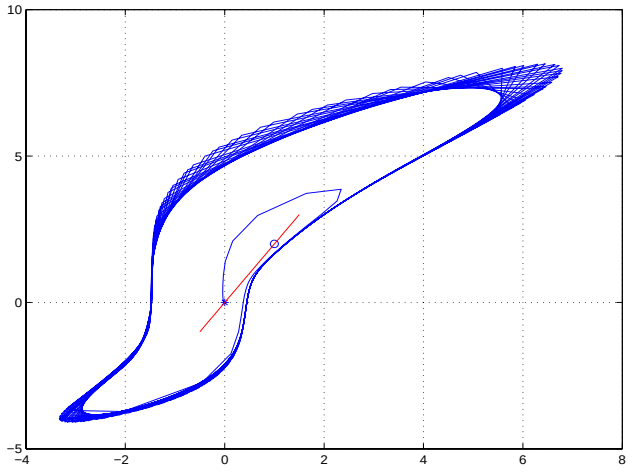
However, the control is again above 0.1,

So let us increase r again by taking

$$r = 500$$

$N = 2,\ M = 3,\ r = 500$





The stability is an issue !

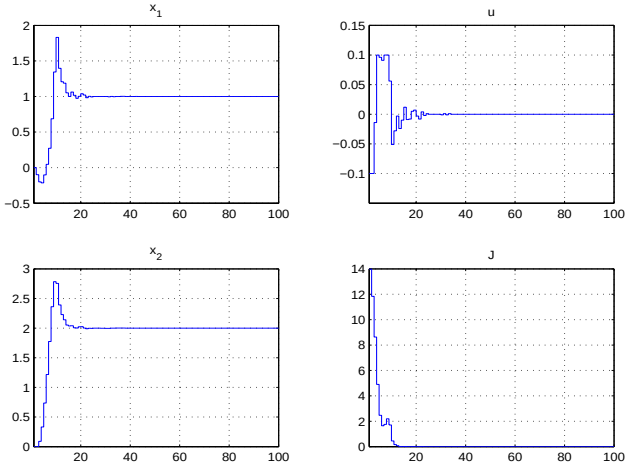
Let us explicitly impose the constraint

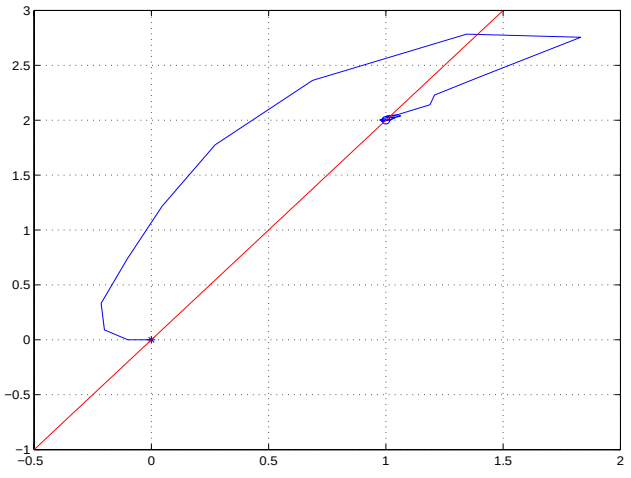
$$\mathbb{U} = [-0.1, 0.1]$$

and test it with the configuration

$$r = 1 \quad ; \quad N = M = 3$$

$N = 3$, $M = 3$, $r = 1$, explicit constraint $|u| \leq 0.1$





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  (Who remember the system's equations ? ! !)

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- In the paper (and related references)
  - Classical formulations with guaranteed stability
  - Contractive formulations
  - Under ideal *optimizer*

The concept of distributed-on-time optimization

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$$\hat{p}(x(t_k)) = \lim_{i \to \infty} p^{(i)}$$

or using some stopping $\varepsilon > 0$:

$$\hat{p}(x(t_k)) = p^{(\imath_\varepsilon(x(t_k)))}$$

where $\imath_\varepsilon(x(t_k)) \in \mathbb{N}$ is the number of iterations that generally depend on $x(t_k)$.

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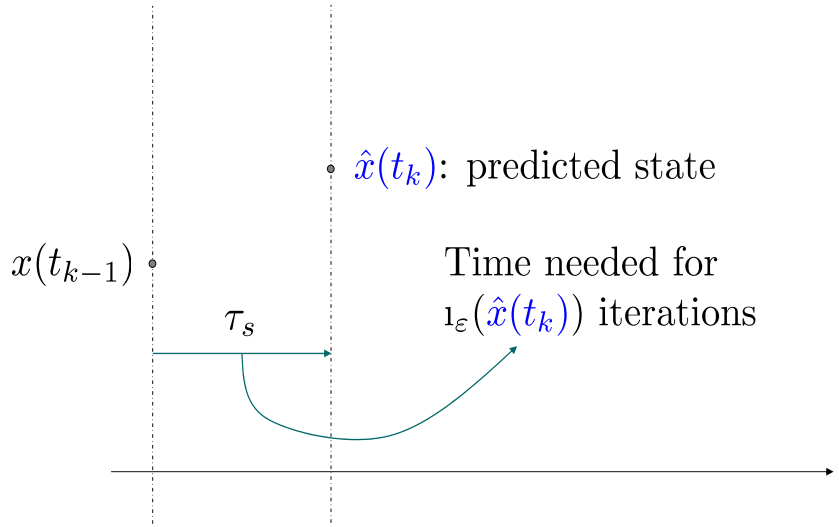

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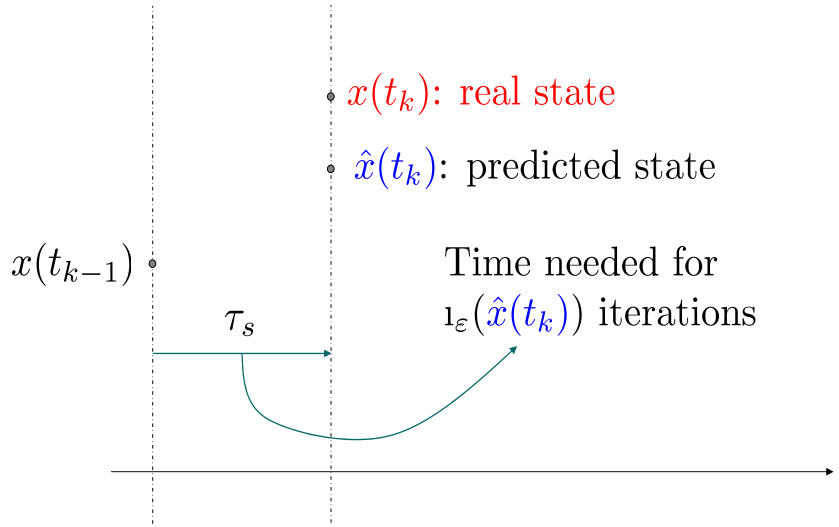

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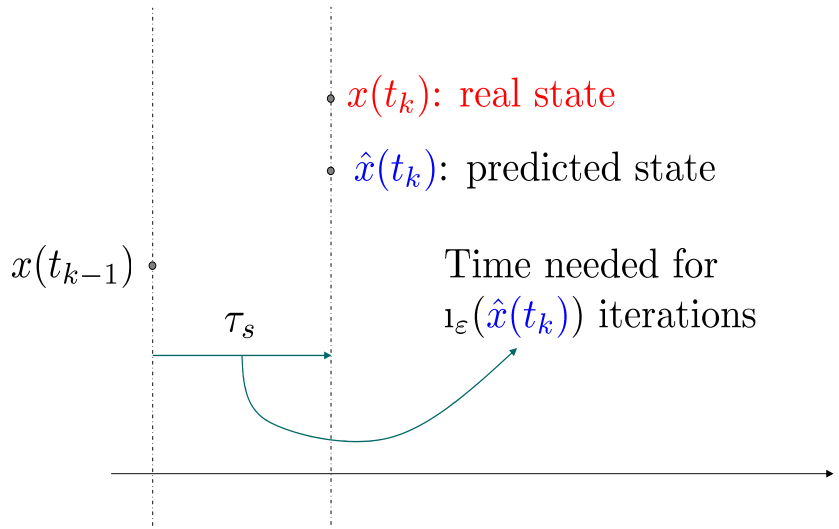

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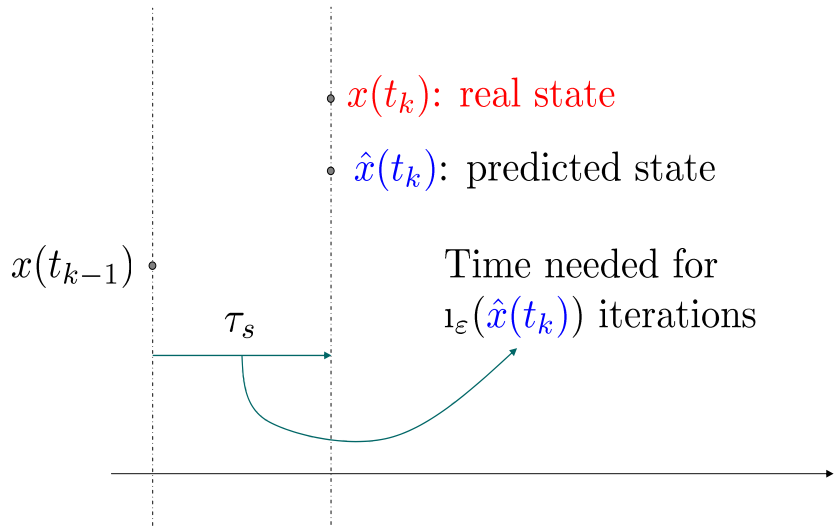

- $\tau_s \geq \imath_\varepsilon(x(t_k)) \times \tau_{\text{iteraton}}$
- $\|x(t_k) - \hat{x}(t_k)\|$ increase with τ_s due to model uncertainties & disturbances

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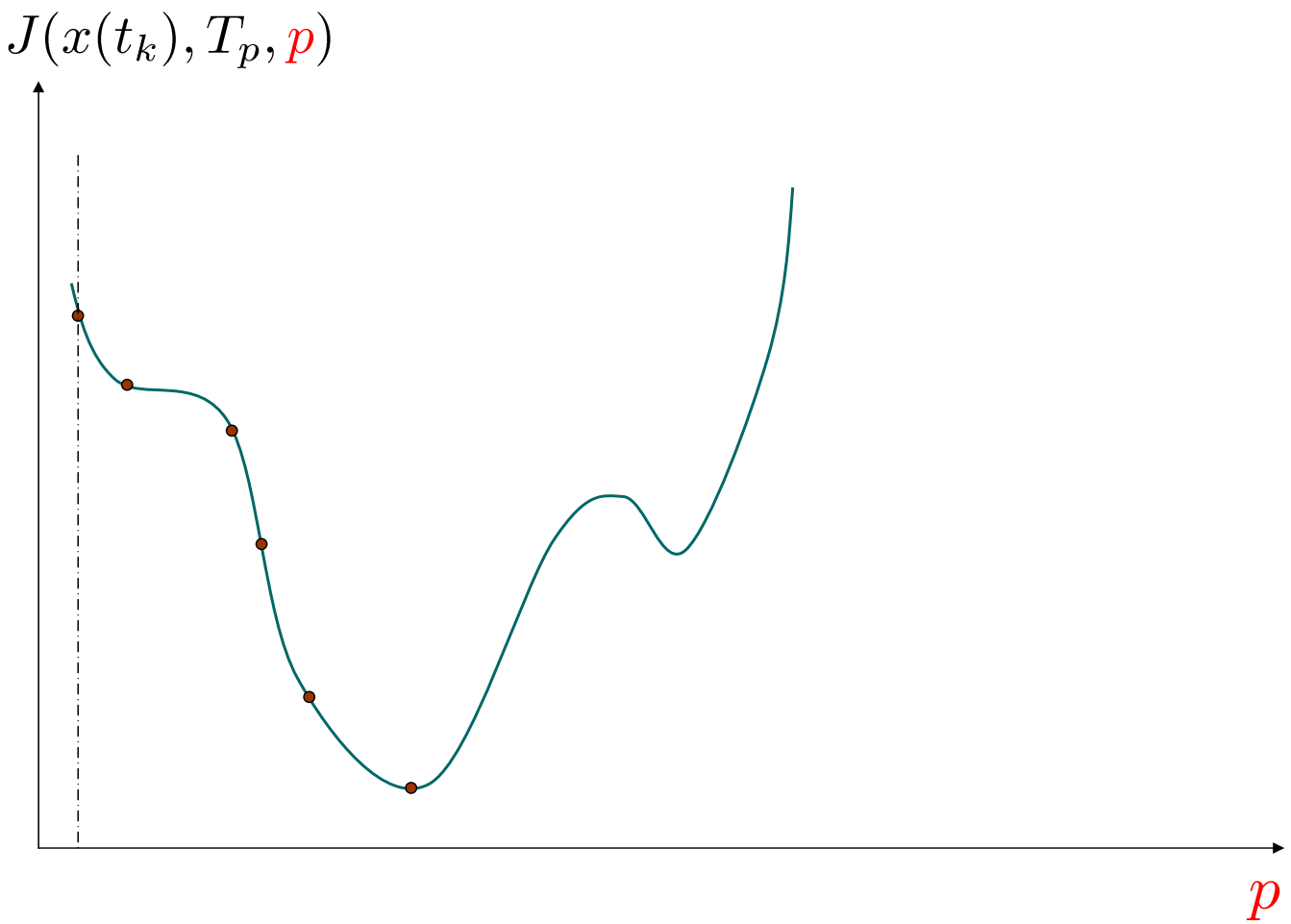
- $\hat{x}(t_{k+1})$ is the predicted state.

A more realistic approach

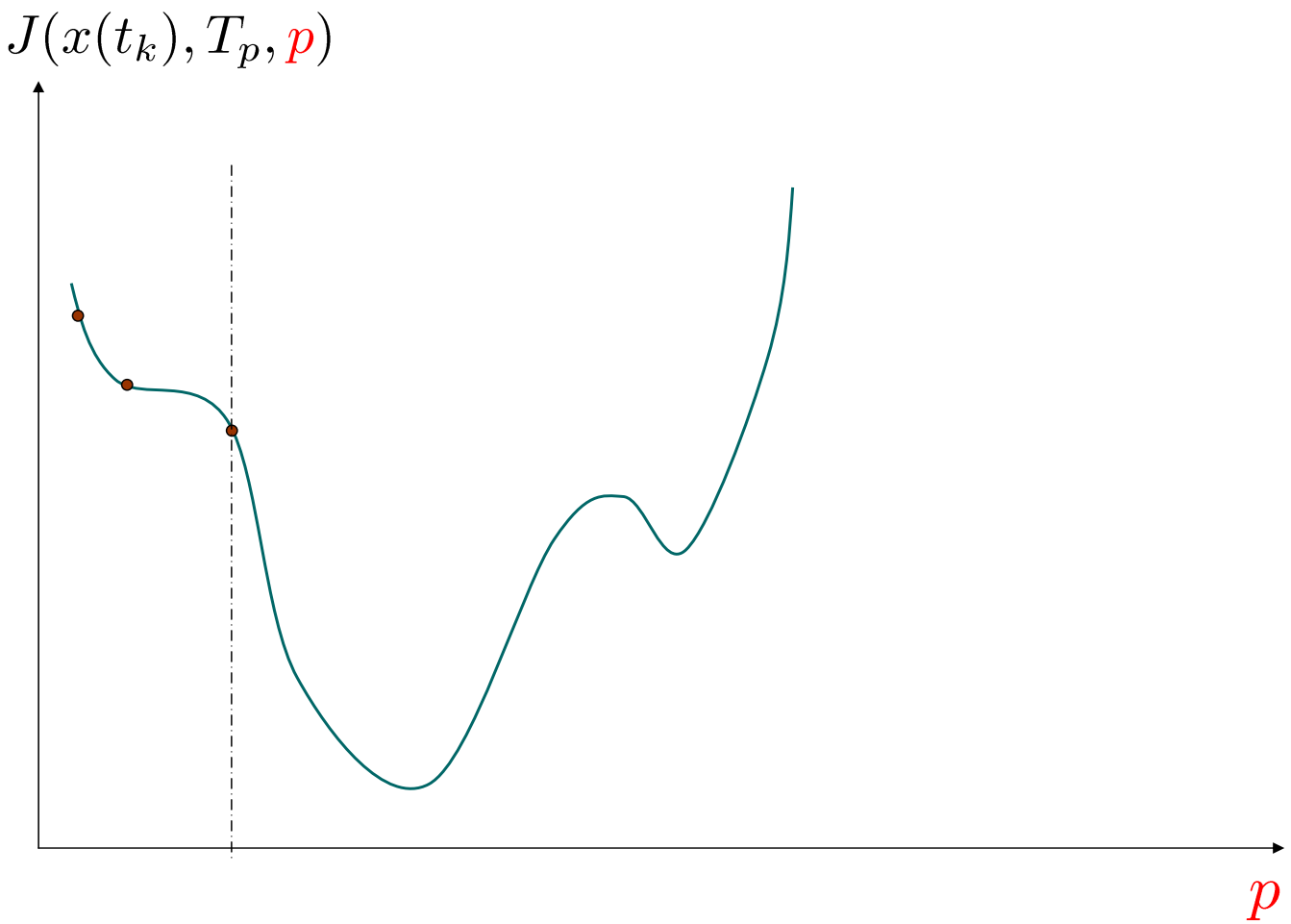
- Choose a sampling period τ_s
- Deduce a maximum number of iterations $q \in \mathbb{N}$ [compatible with τ_s]
- Distribute the optimization over the system real life-time :

$$\begin{aligned} x(t_{k+1}) &= X(\tau_s, x(t_k), u^1(p(t_k))) \\ p(t_{k+1}) &= \mathcal{S}^{(q)}(p^+(t_k), \hat{x}(t_{k+1})) \end{aligned}$$

- p^+ is defined by the translatability of the control parametrization.
- $\hat{x}(t_{k+1})$ is the predicted state.
- Stability is a more involved issue than in classical formulations



Case $q = 2$



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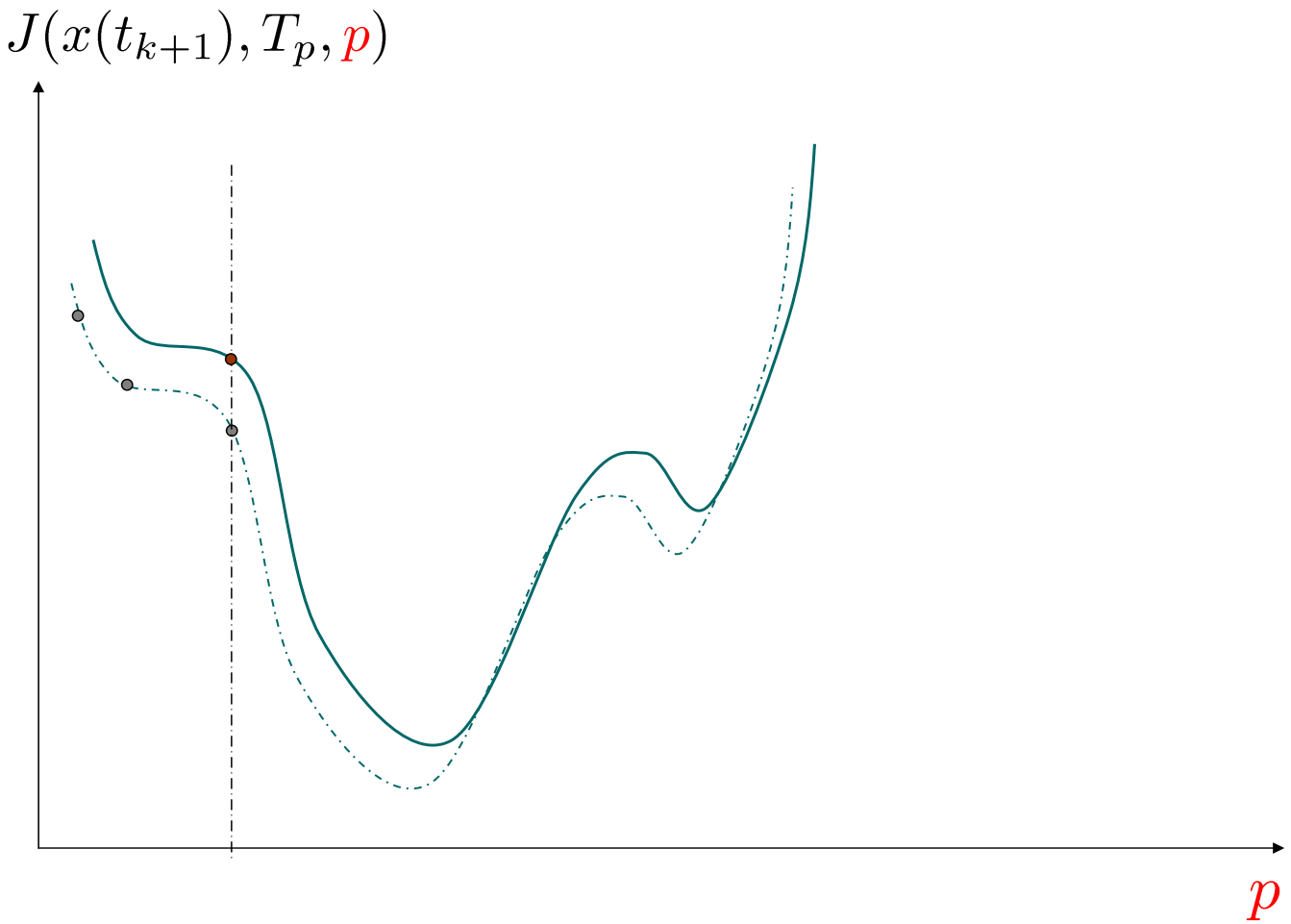

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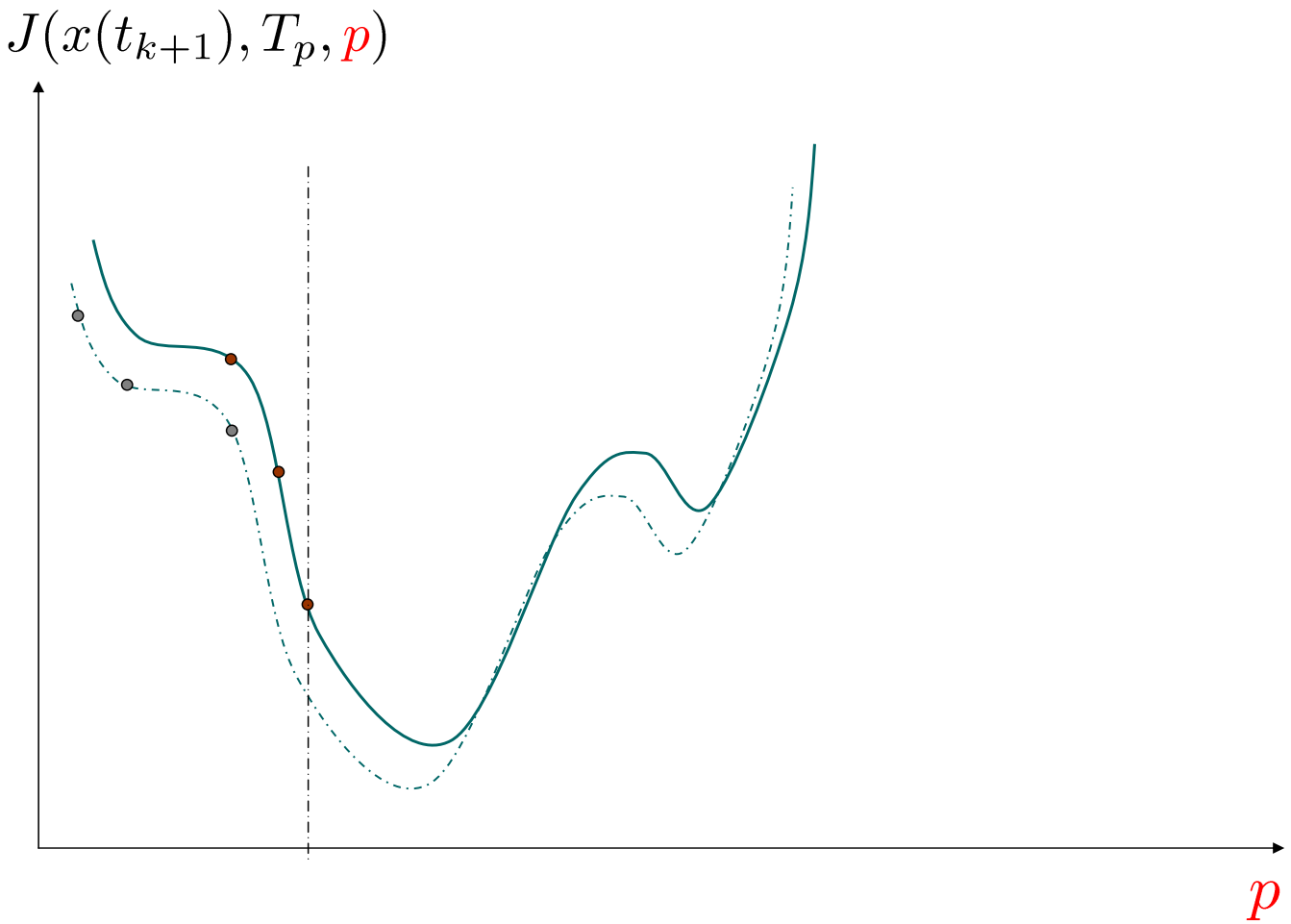

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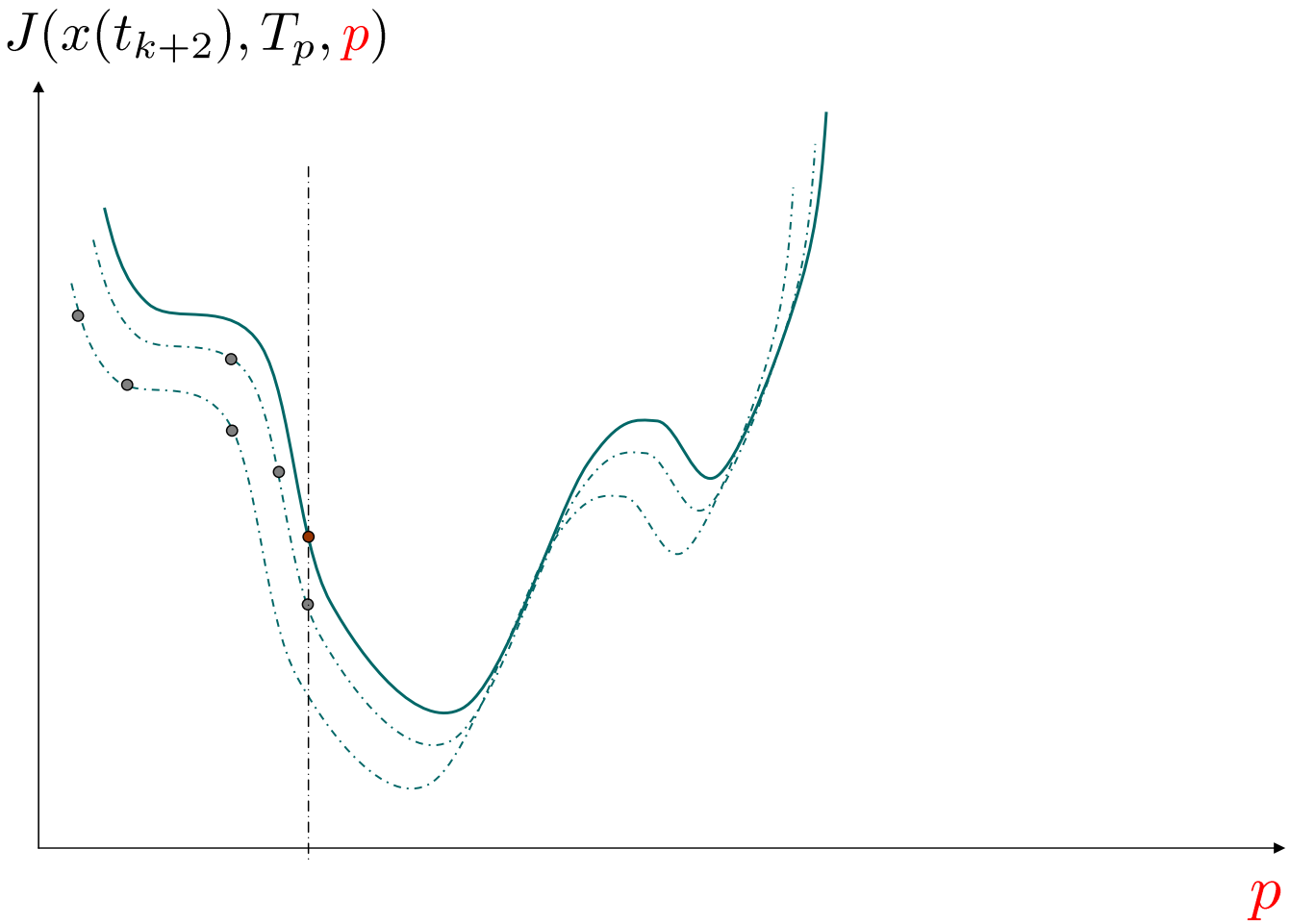
gipsa-lab



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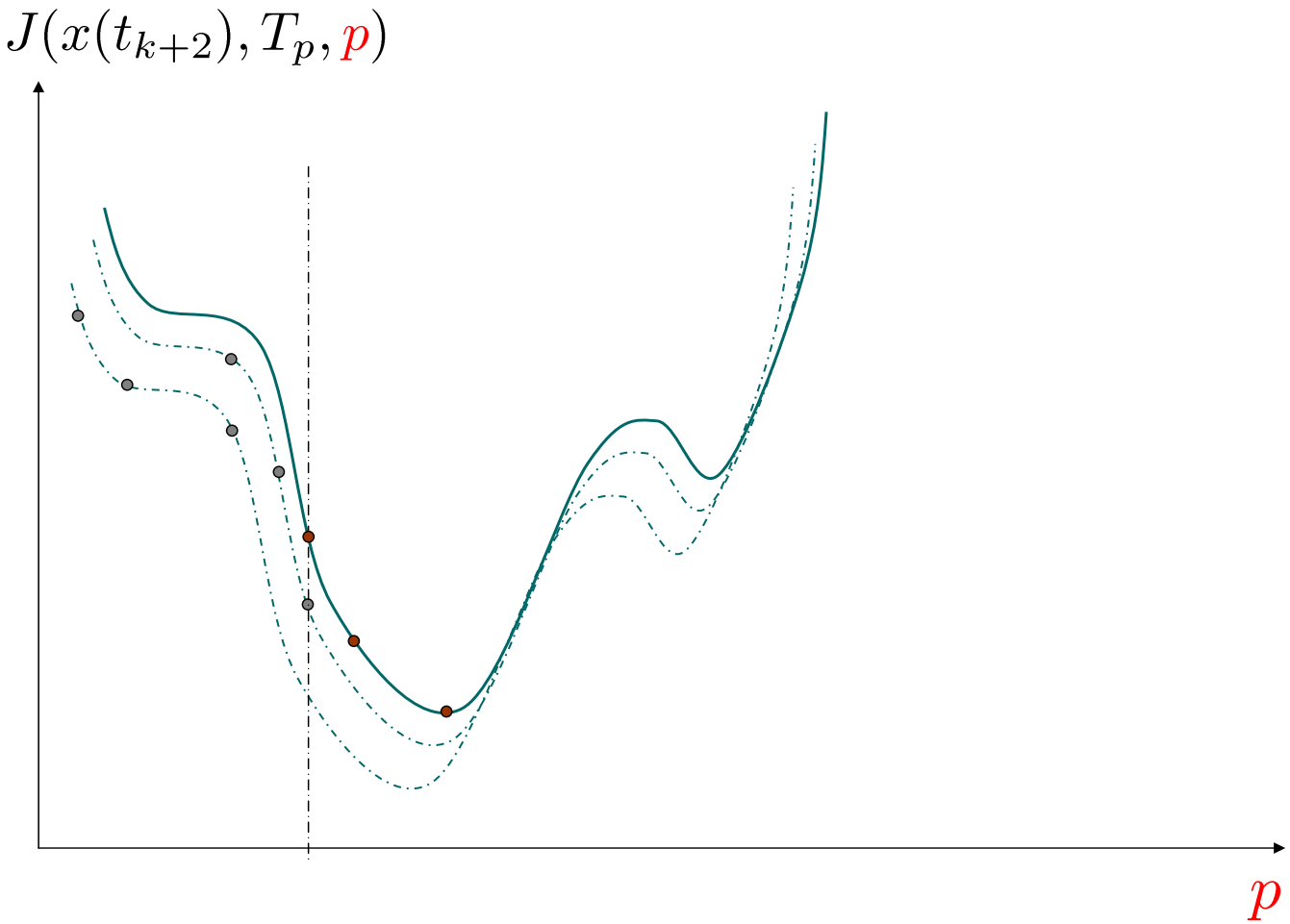

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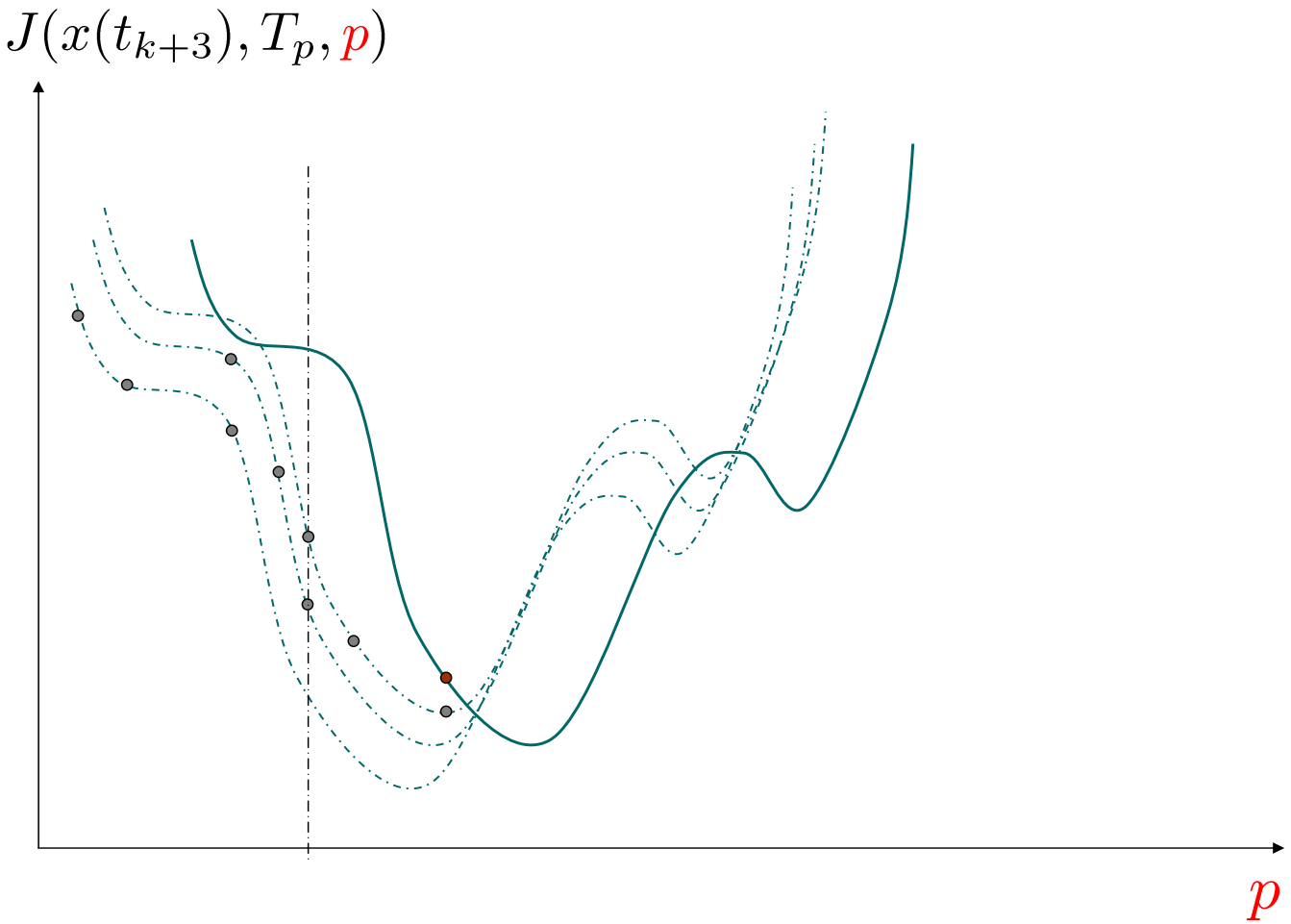

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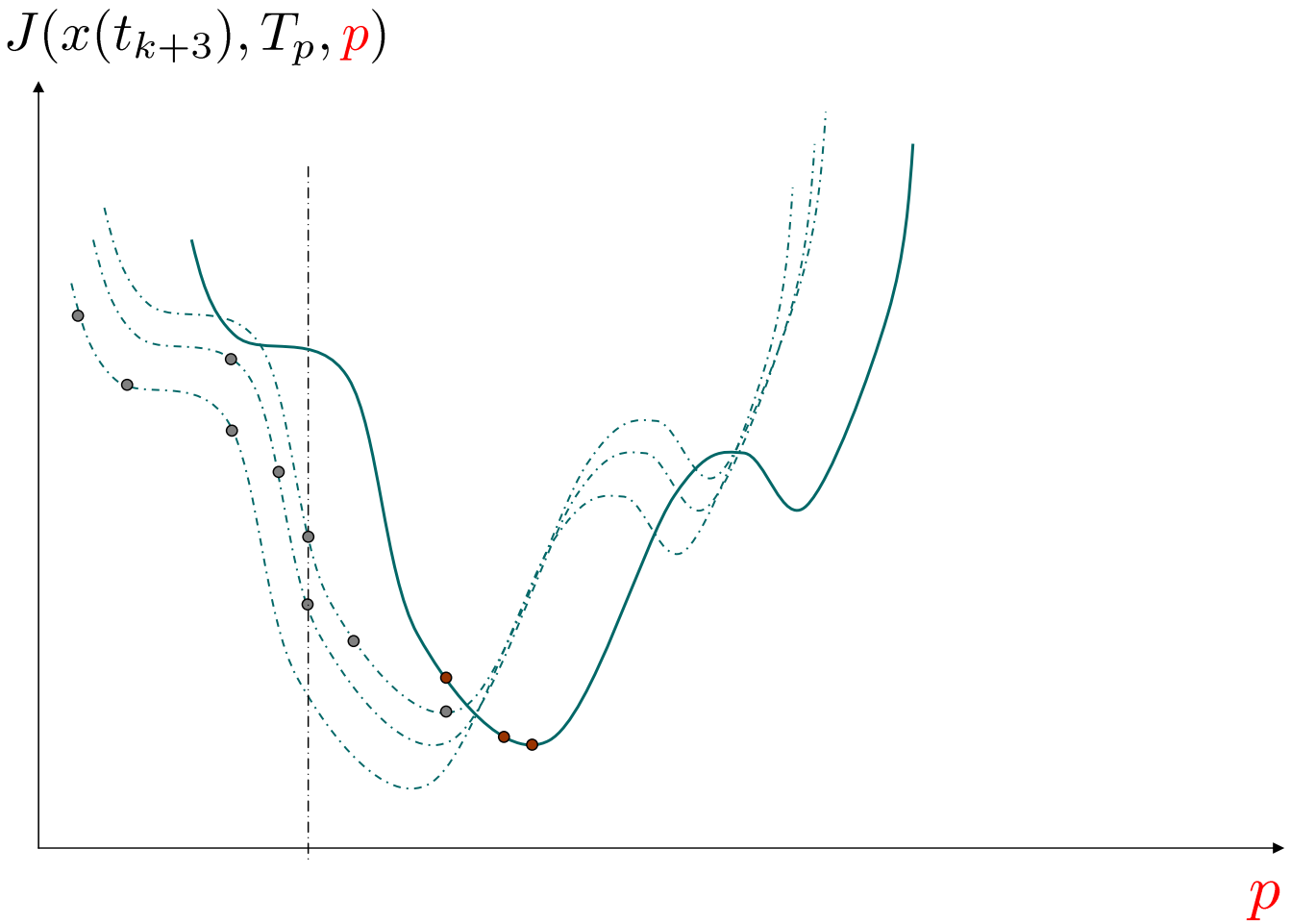

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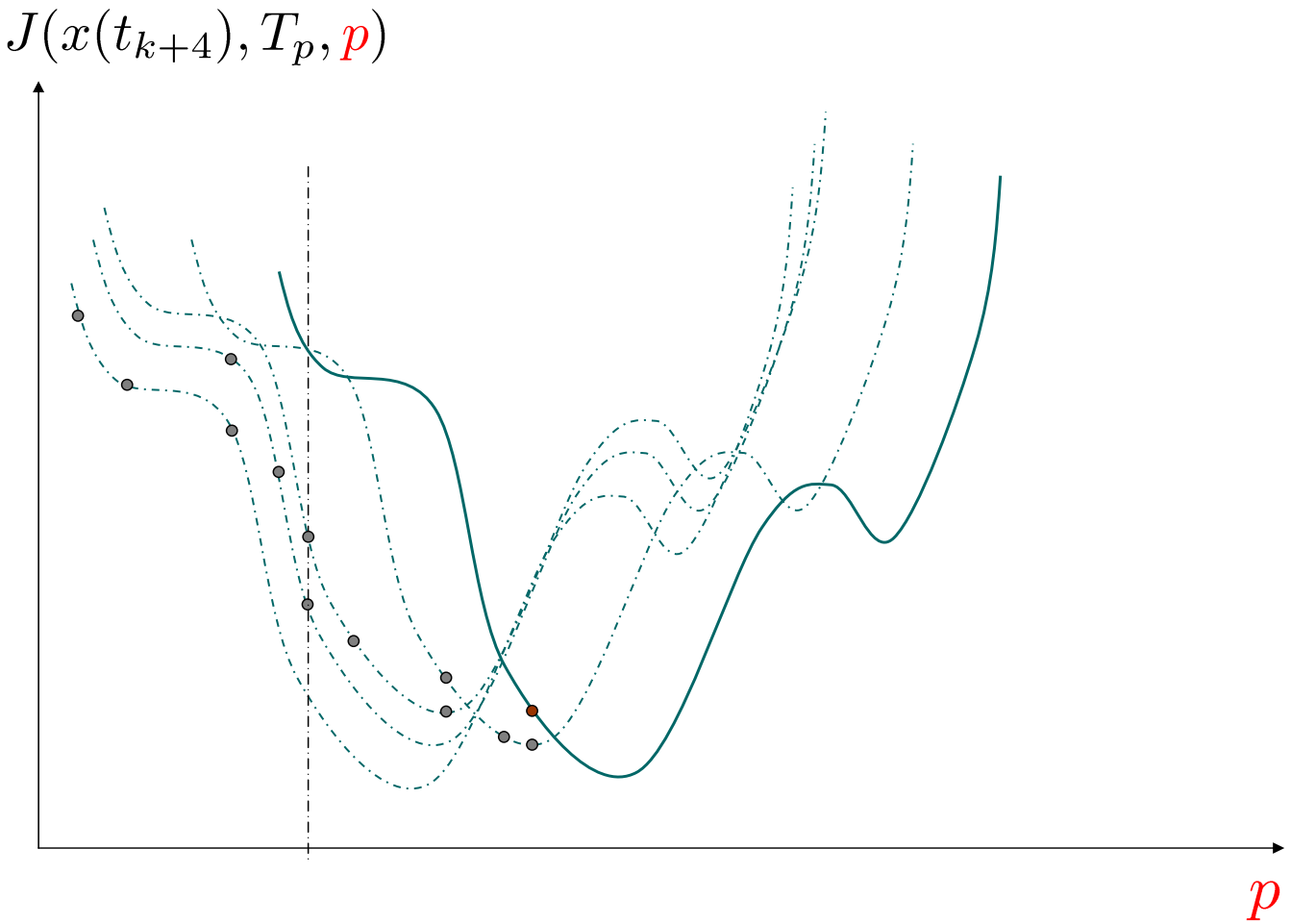

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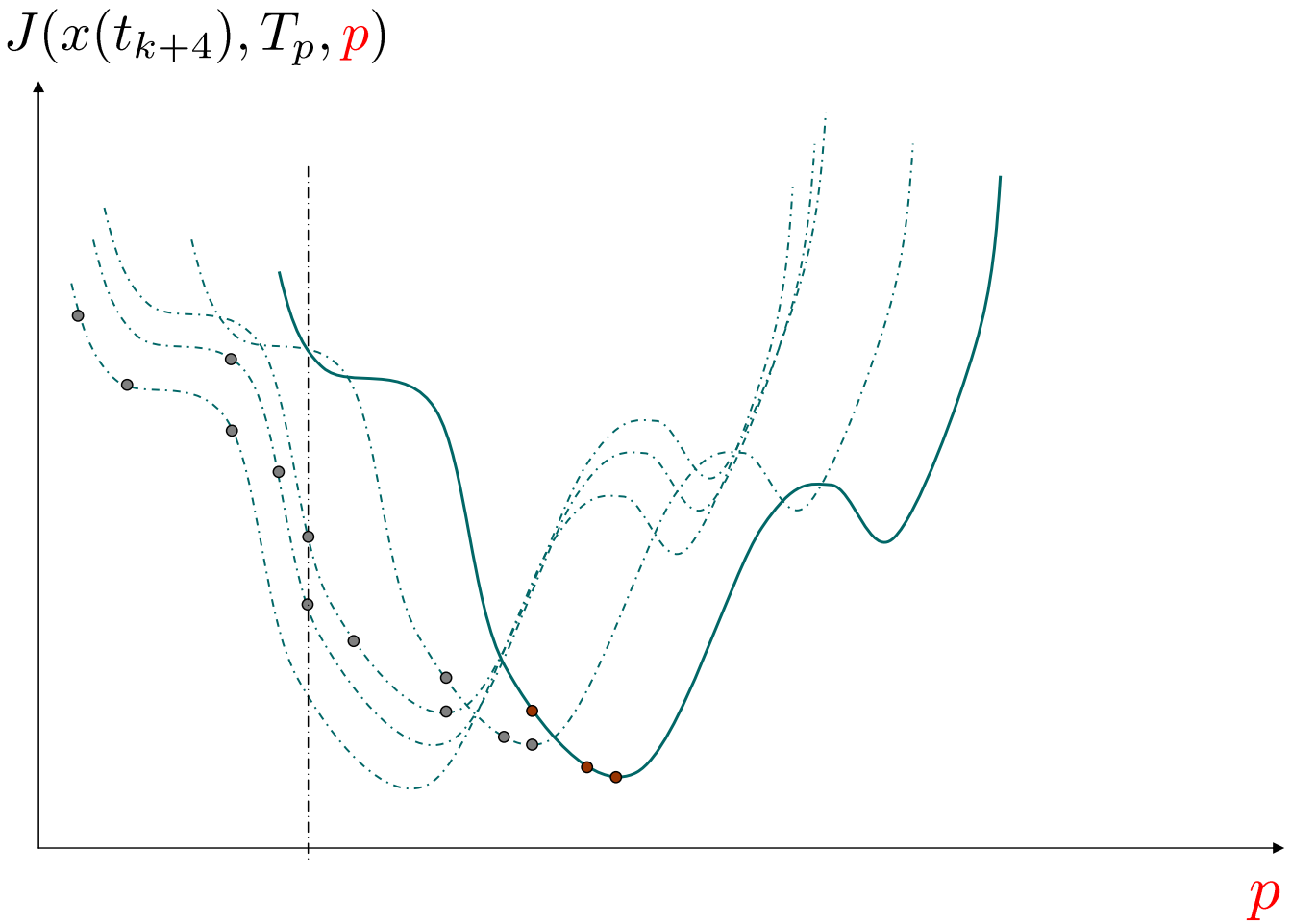

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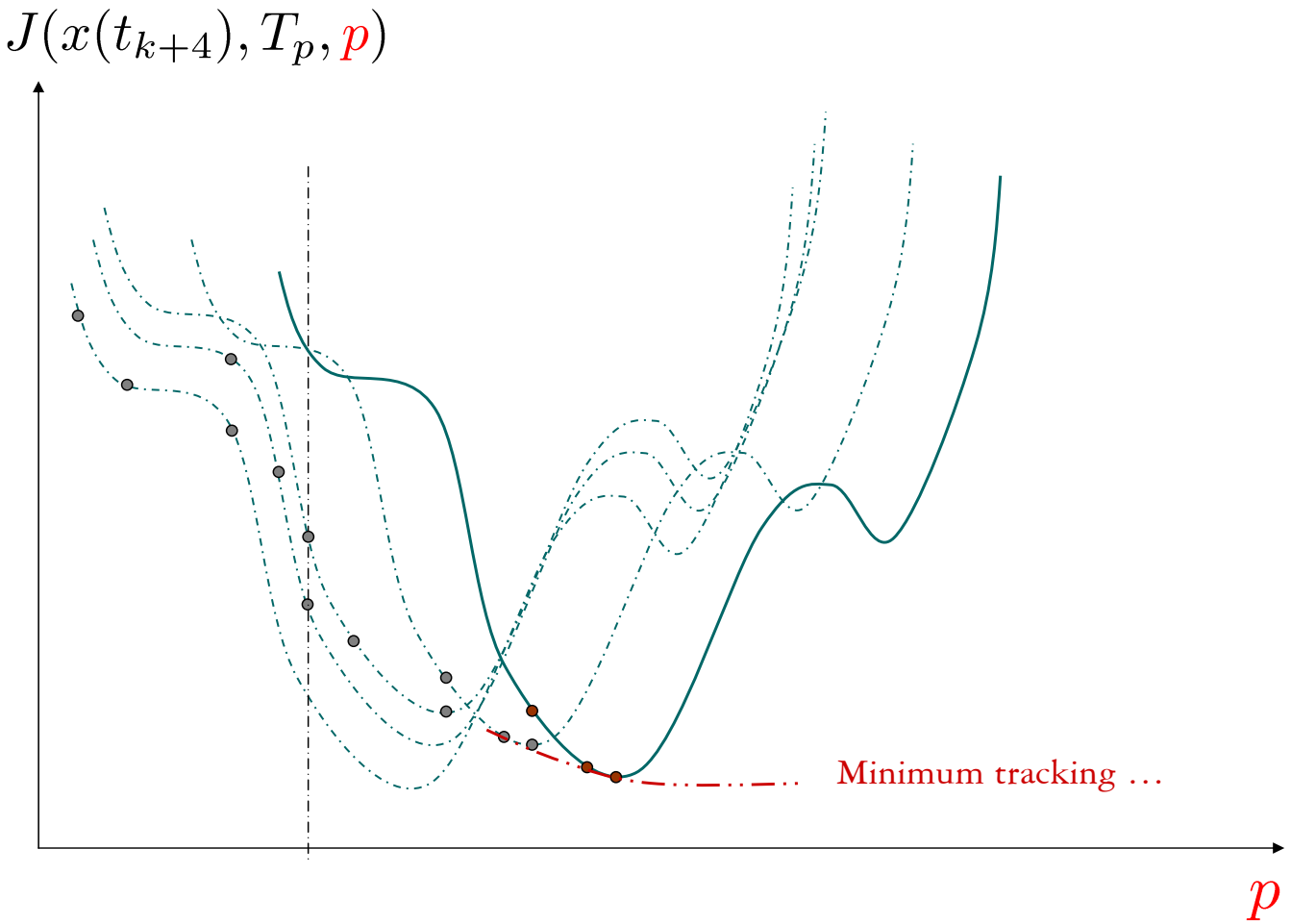

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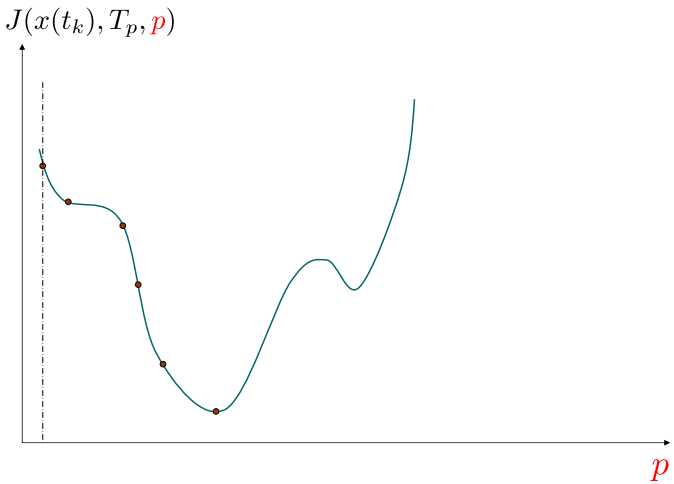

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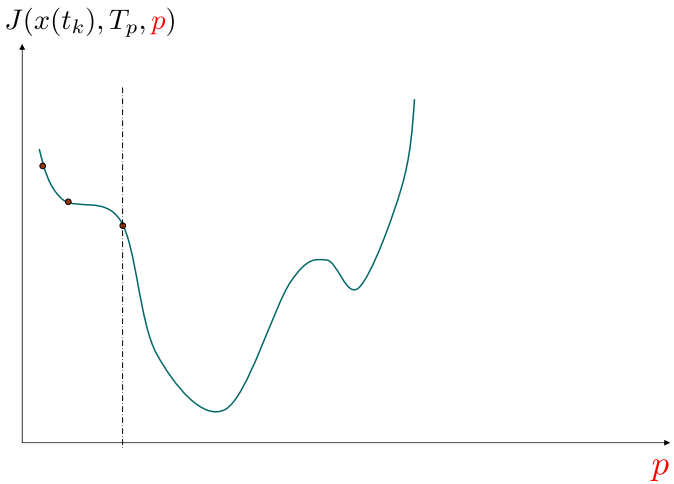
2. The iteration is efficient
3. No finite escape time in the transient phase
4. The original RH formulation is stabilizing



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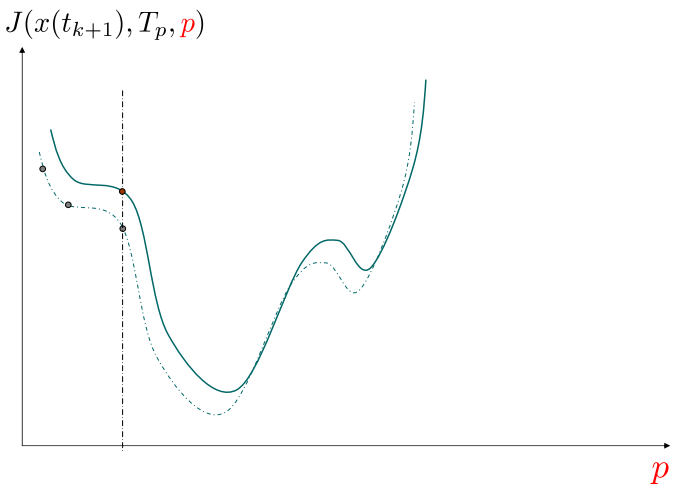

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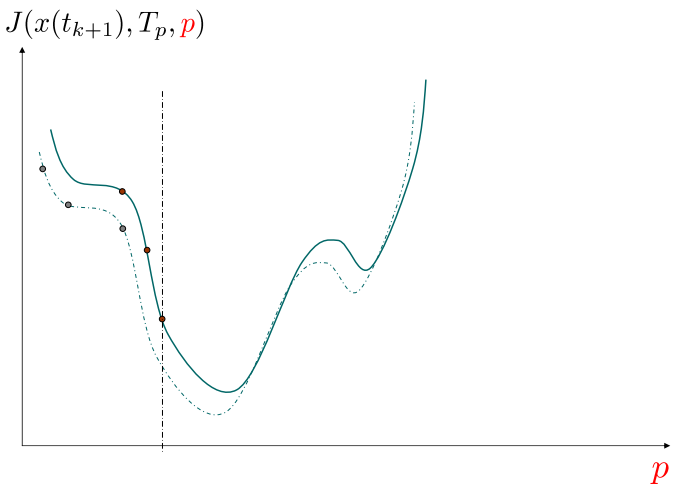

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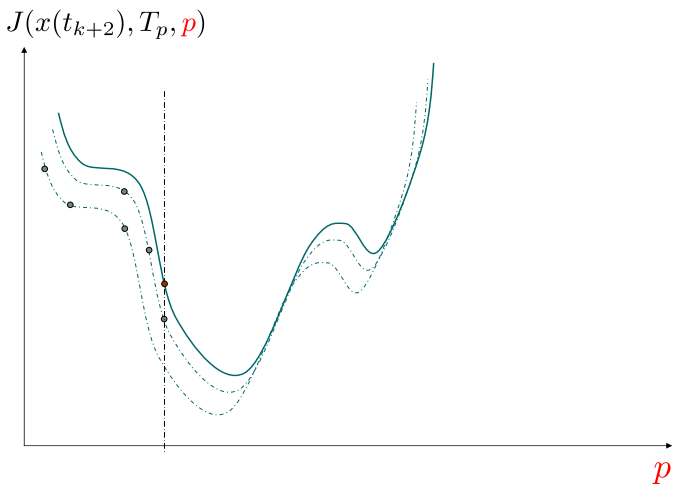

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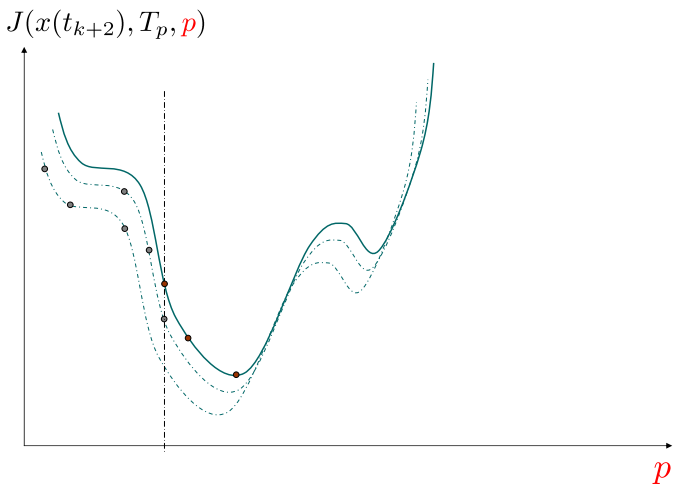

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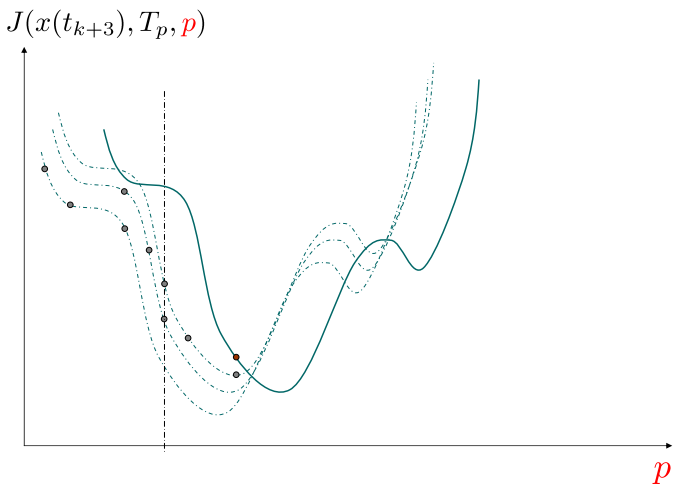

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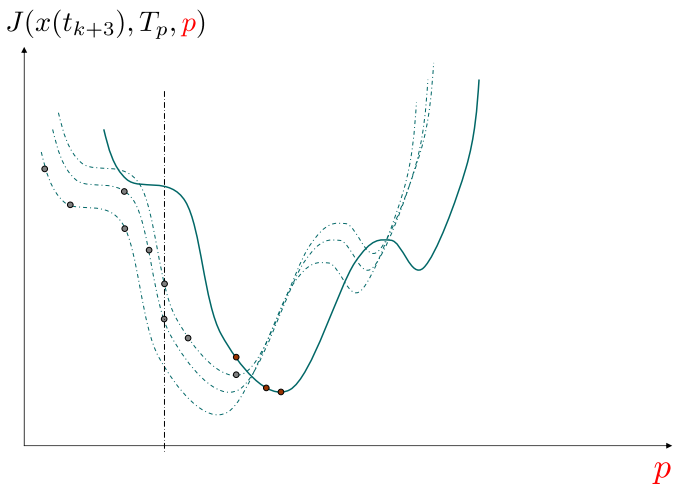

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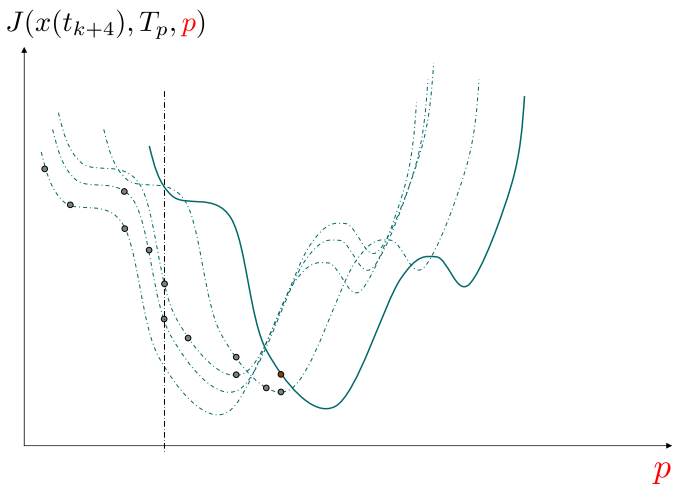

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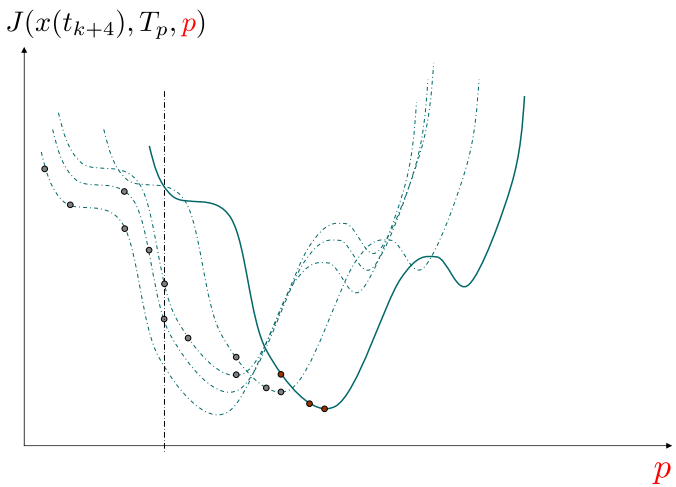
gipsa-lab



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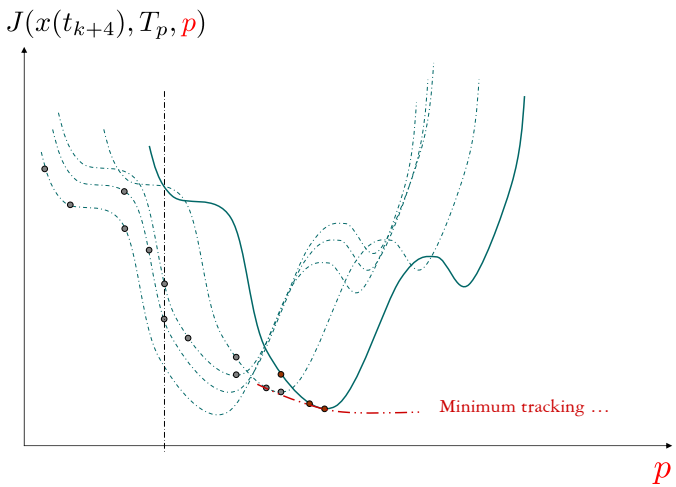

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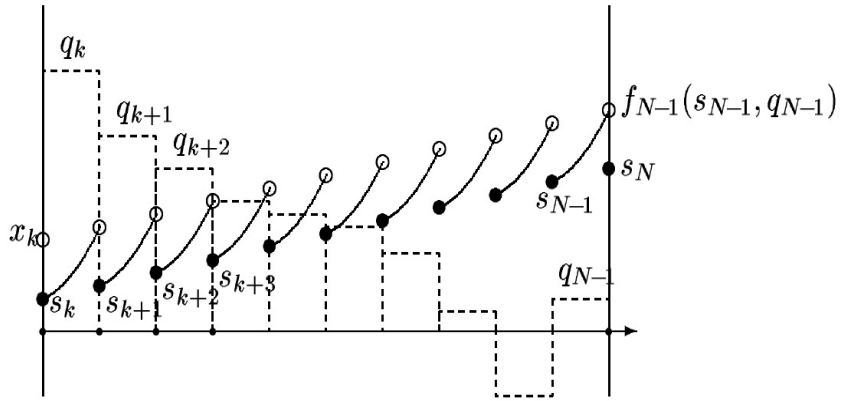
The iterative process $\mathcal{S}$

$$\begin{aligned} p^{(i+1)} &= \mathcal{S}(p^{(i)}, x(t_k)) \\ p^{(0)} &= p_0 \in \mathbb{P} \end{aligned}$$

1. Multiple shooting approach [Diehl et al. 2005]
2. Continuation (homotopy) approach [Ohtsuka 2004]
3. Parametric approach

The multiple shooting approach

[Diehl et al. 2005]



Unknown

$$y = (\lambda_k, s_k, u_k, \ldots, \lambda_{k+N}, s_{k+N}, u_{k+N})^T$$

Constraints

$$x_k - s_k = 0$$

$$f(s_i, u_i) - s_{i+1} = 0$$

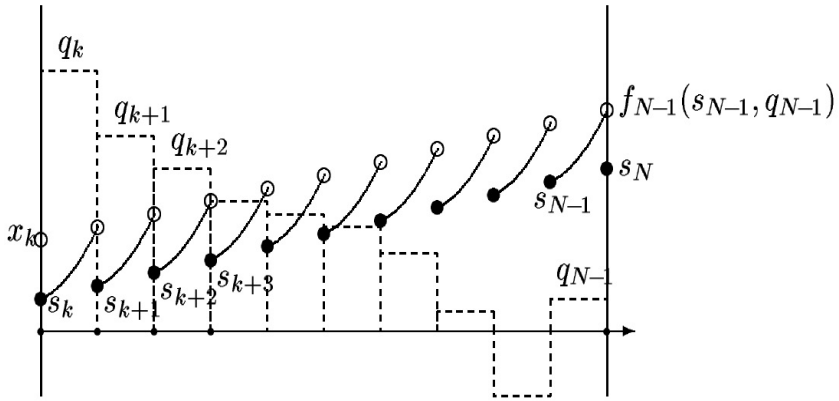
gipsa-lab

The multiple shooting approach

[Diehl et al. 2005]

Karush-Kuhn-Tucker optimality condition :

$$\nabla_y \mathcal{L}^k(y) = 0$$



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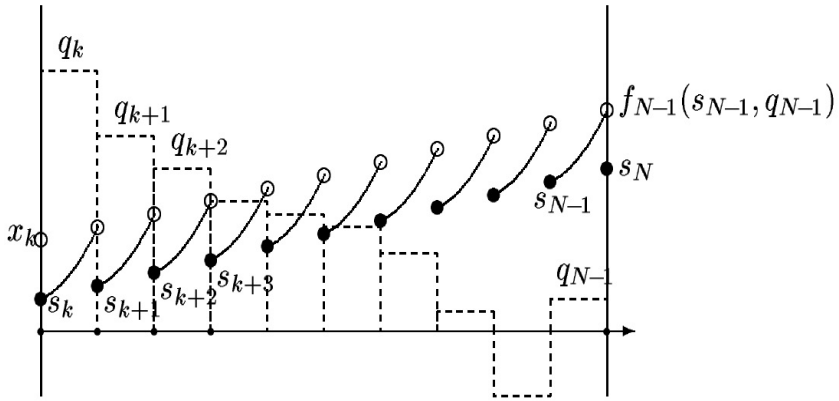
$$\nabla_y \mathcal{L}^k(y) = 0$$

At each instant

$$\nabla_y \mathcal{L}^k(y) + J^k(y) \cdot \Delta y = 0$$



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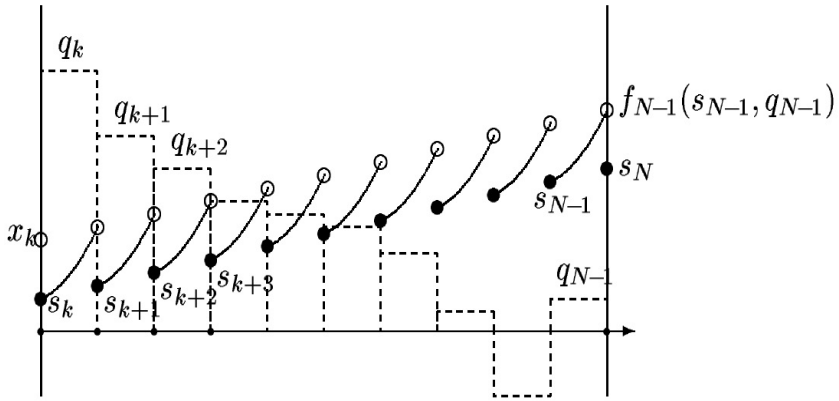

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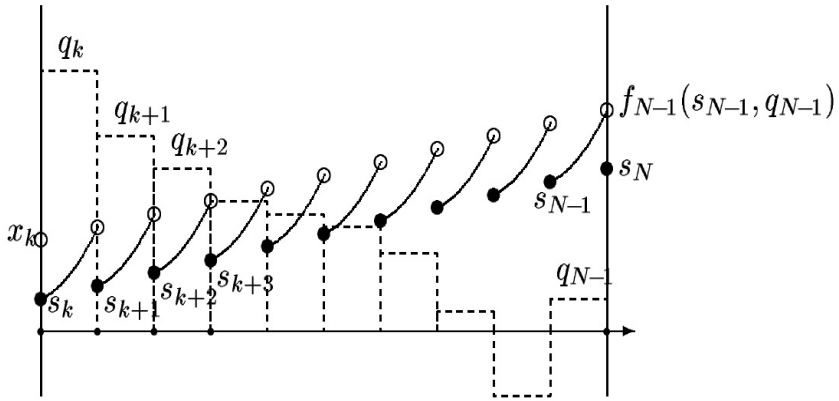
- $J^k(y)$ independent of $x(k)$
- transient violation of feasibility



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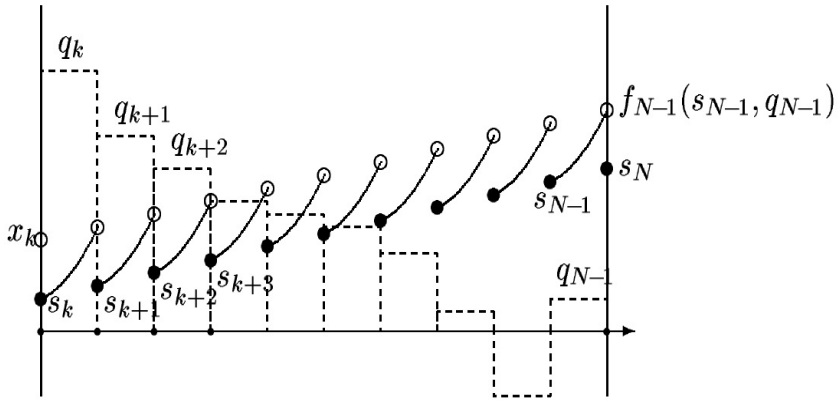

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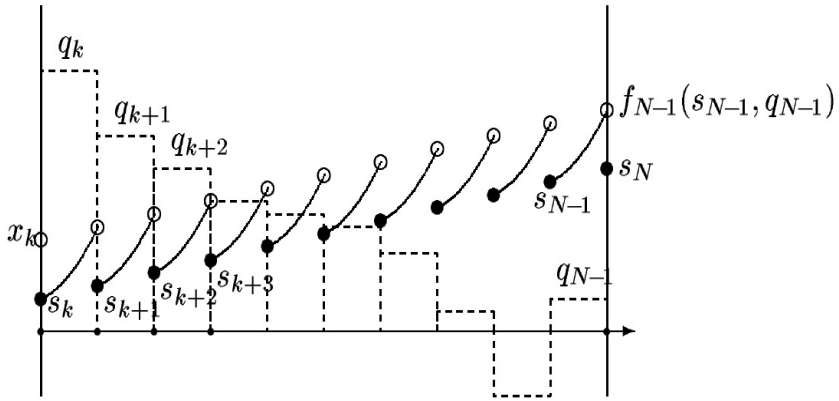
Available software MUSCODII



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gipsa-lab

The continuation approach

[Ohtsuka et al. 2004]

Hamiltonian system

$$\dot{x} = f(x, u)$$
$$\dot{\lambda} = -H_x(x, u, \lambda, \mu)$$

Hamiltonian

$$H(x, \lambda, \mu, u) =$$
$$L(x, u) + \lambda^T f(x, u) + \mu^T C(x, u)$$

Unknown

$$U := \left(u_0^T, \mu_0^T, \ldots, u_{N-1}^T, \mu_{N-1}^T\right)^T$$

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Optimality condition

$$F(U(t), x(t)) = 0$$

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By imposing

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Krylov subspace approach

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$$F_U W + F_x w \approx \frac{F(U + hW, x + hw) - F(U, x)}{h}$$

gipsa-lab

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- Need initialization
- Need differentiability

The parametric approach

Fact 1

Simple OL param.

$\downarrow$

Complex CL behavior

The parametric approach

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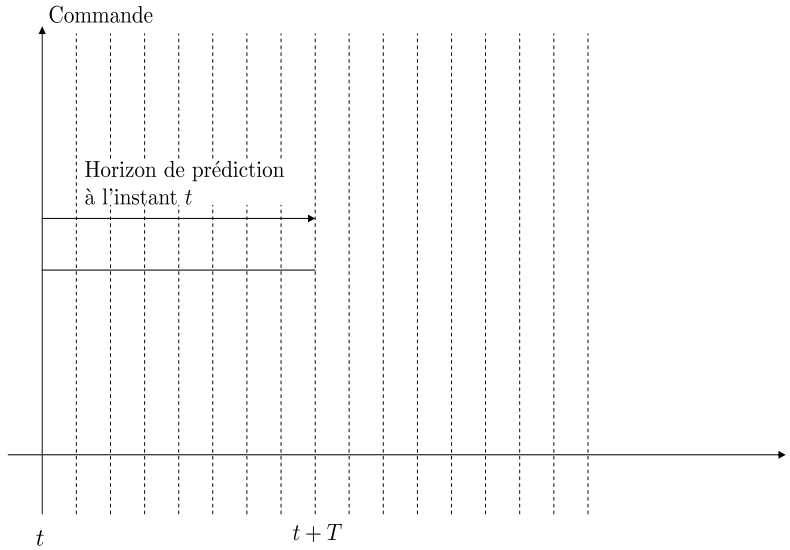
Simple OL param.

$\downarrow$

Complex CL behavior



The parametric approach

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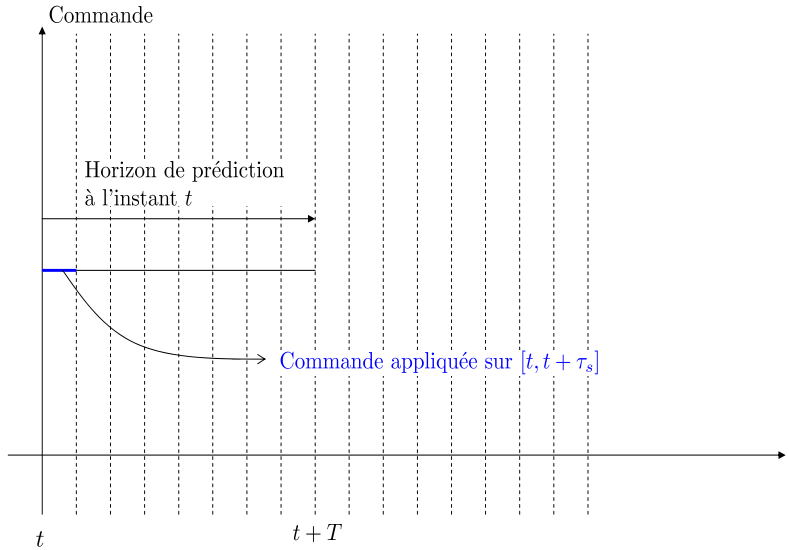
Complex CL behavior



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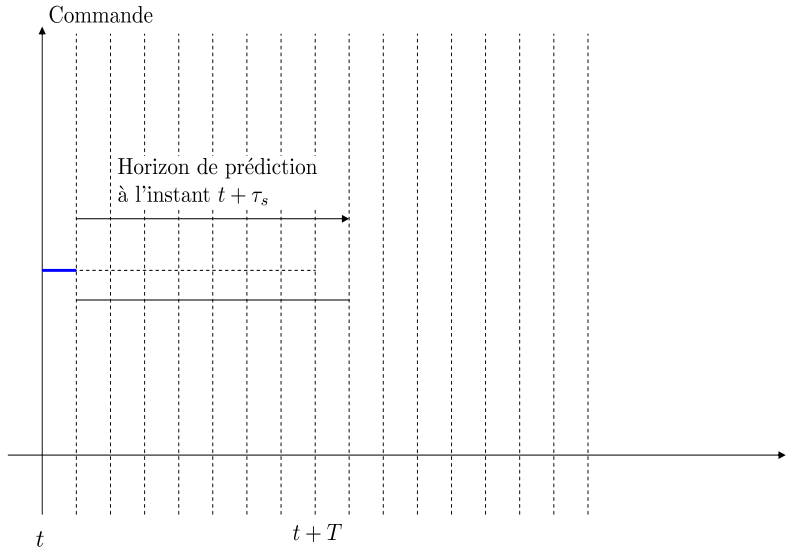

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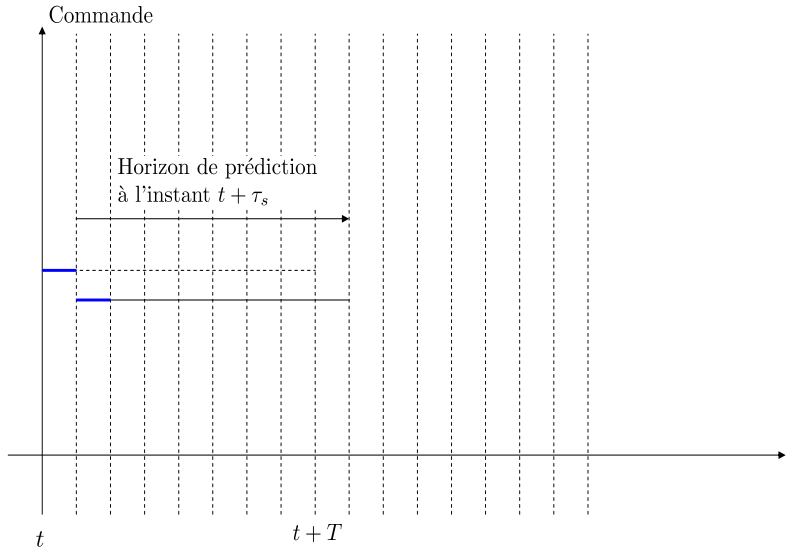

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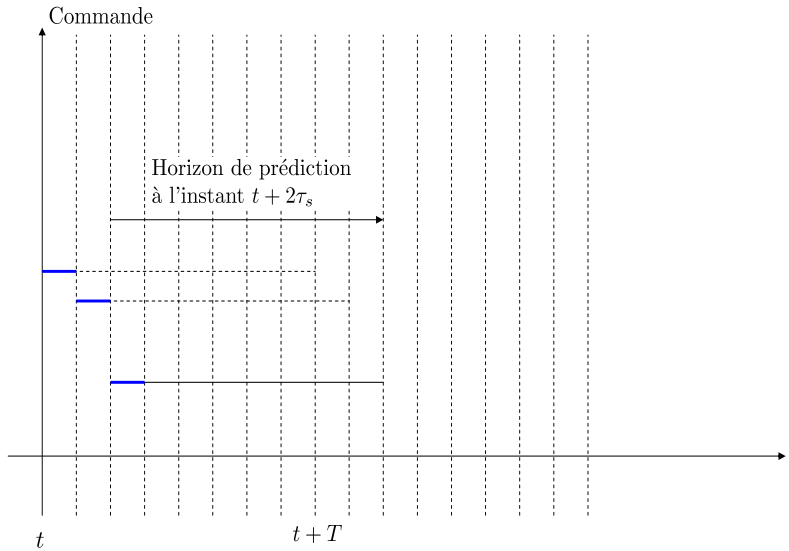

The parametric approach

Fact 1

Simple OL param.

↓

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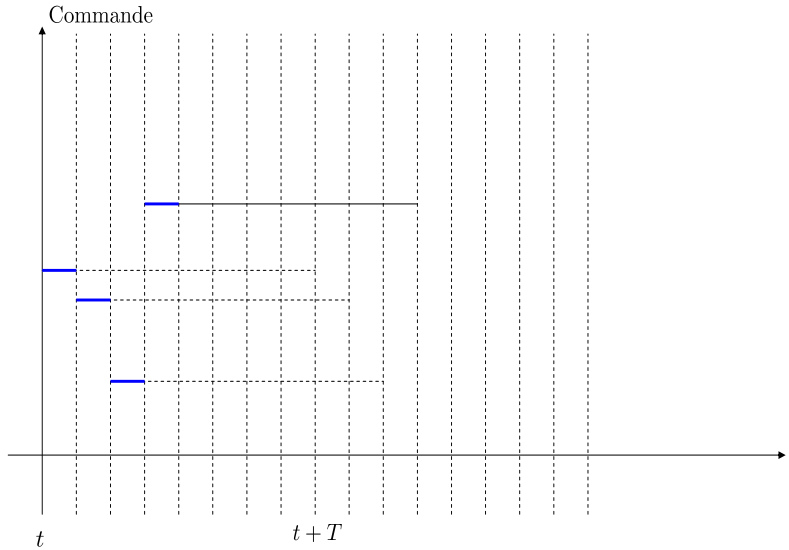

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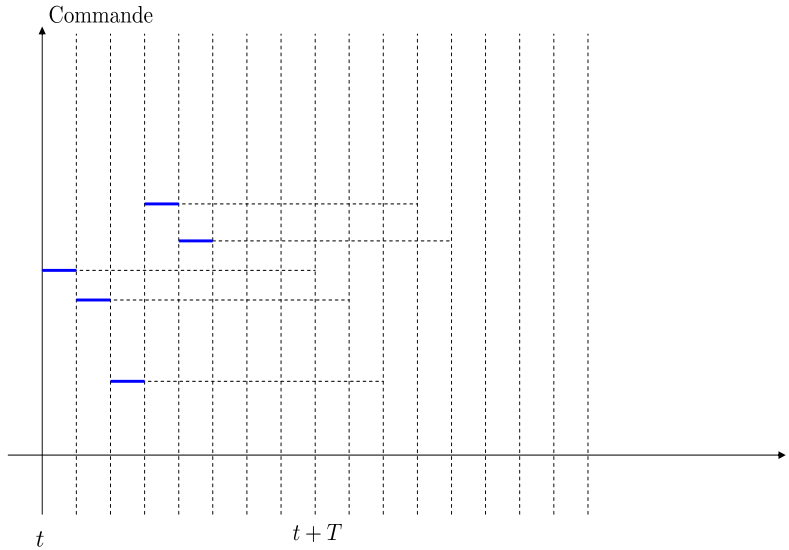

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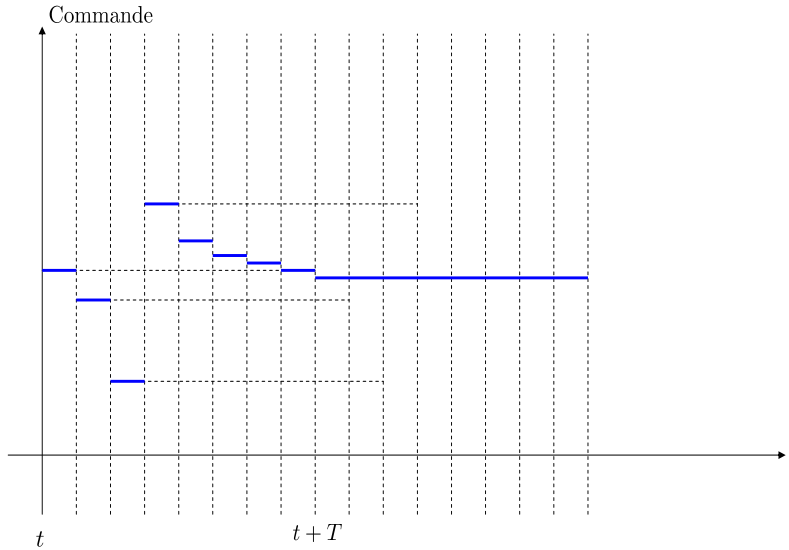

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$\downarrow$

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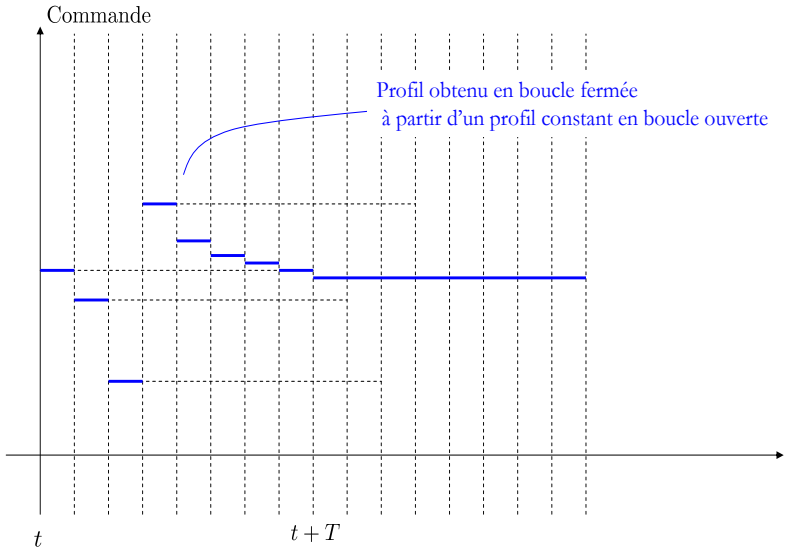


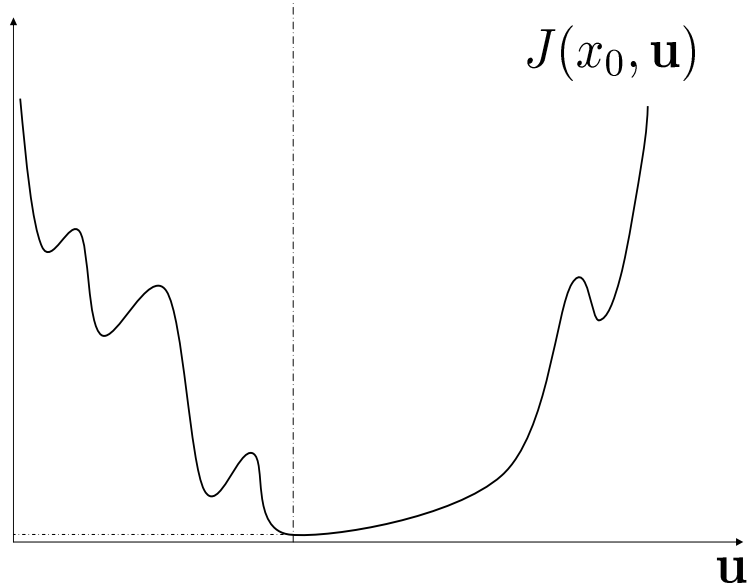
The parametric approach

Fact 2

Solve exactly a sub-optimal formulation

could be better than

Solve loosely the exact problem



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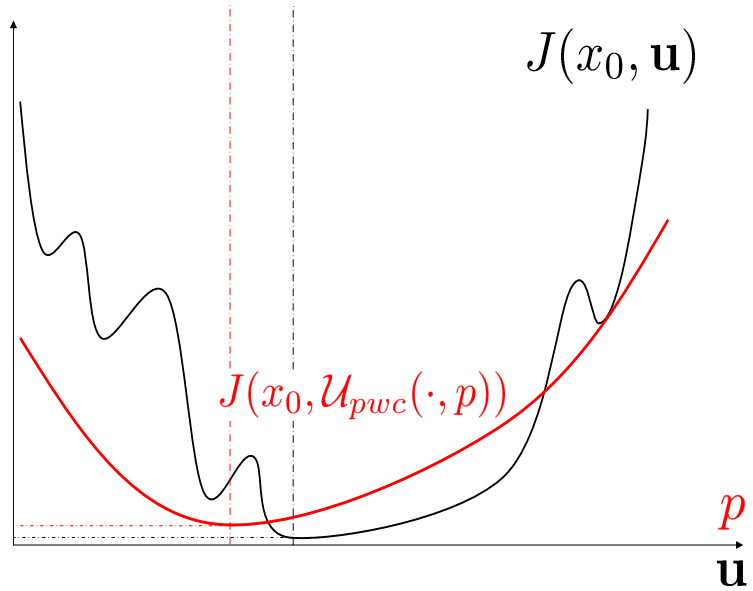

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The parametric approach

Fact 3

Constraints can often be embedded in the control parametrization

↓

Optimization improves performance under guaranteed stability and constraints fulfilment

The parametric approach

Fact 4

Reduced dimensional and well posed optimization problems

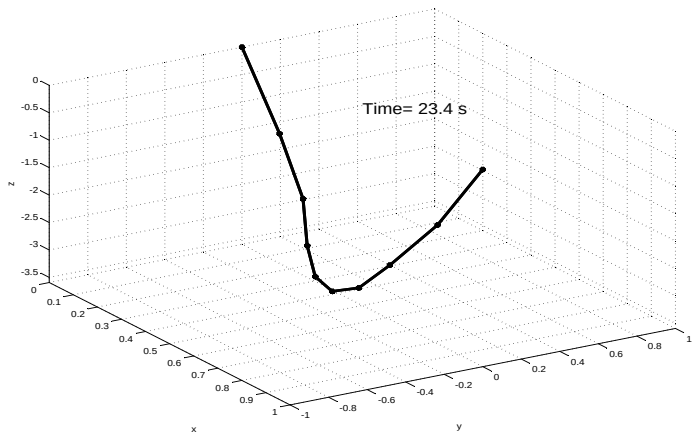
Render efficient

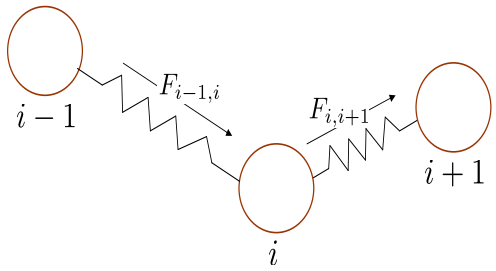
A family of simple, non smooth and memoryless algorithms that would be out of scope for large scale problems

Problème	n_p	temps de calcul (s)
Interception en temps minimal	1	18×10^{-3}
Stabilisation des systèmes chaînés	1	$< 10^{-3}$
Stabilisation d'un avion à décollage vertical	2	$< 10^{-3}$
Simple pendule inversé avec chariot	1	< 0.1
Double pendule inversé avec chariot	2	< 0.25
Stabilisation de satellite en mode défaillant	2	< 0.01

Non damped elastic chain

Non damped elastic chain





$$\dot{x}_i = \frac{1}{m}\Big(F_{i,i+1} - F_{i-1,i}\Big) + g \quad ; \quad i = 1, \ldots, N$$

$$\dot{x}_{N+1} = u \qquad \dot{x}_i \in \mathbb{R}^3$$

$$F_{i,i+1} = D\left(1 - \frac{L}{\|x_{i+1} - x_i\|}\right)(x_{i+1} - x_i)$$

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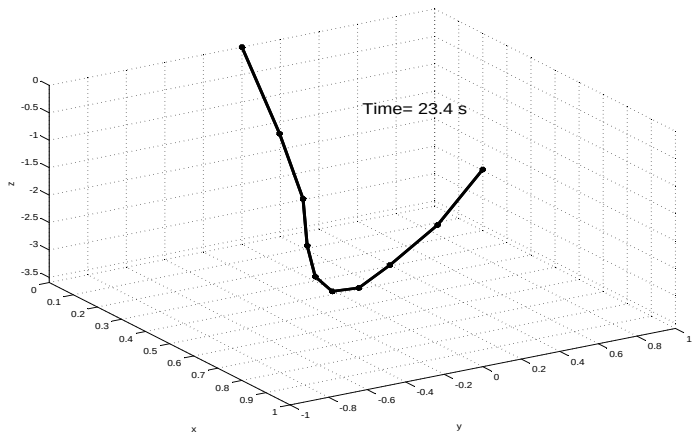

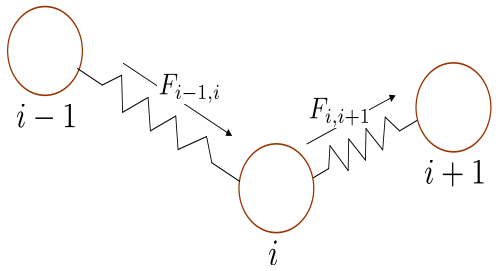

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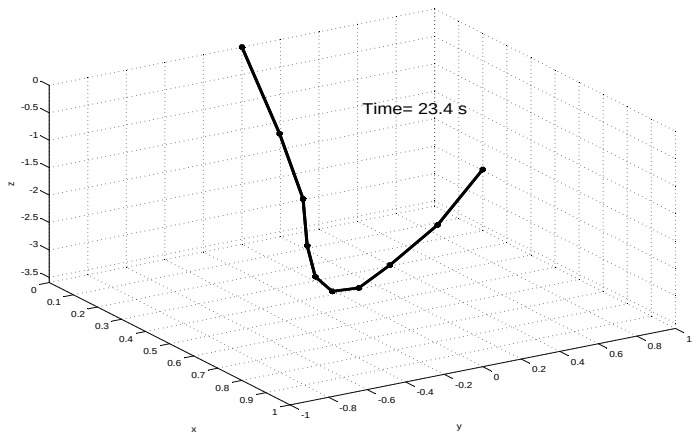
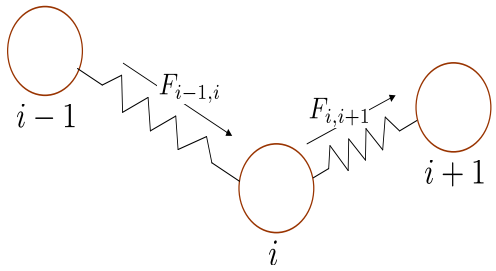
- 10 balls $\longrightarrow$
  - $(9 \times 3 \times 2) - 3 = 51$ states
  - 3 Controls

Non damped elastic chain





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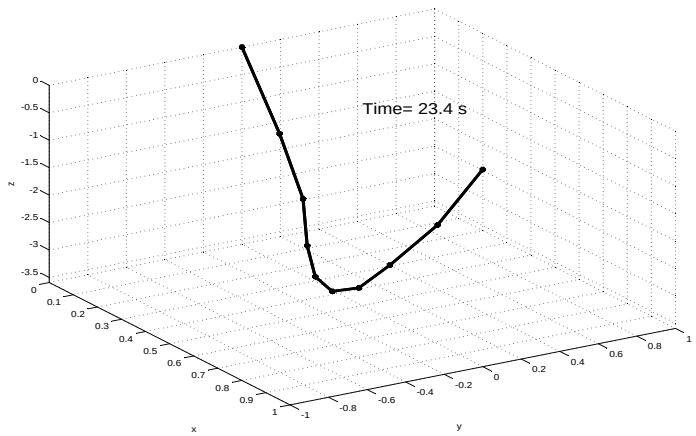

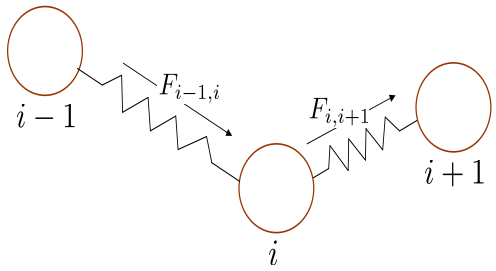

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- State constraints

  $P_{N+1} \in P_f + [-\Delta P_{max}, +\Delta P_{max}]^3$
  $P_{N+1} \longrightarrow P_f$

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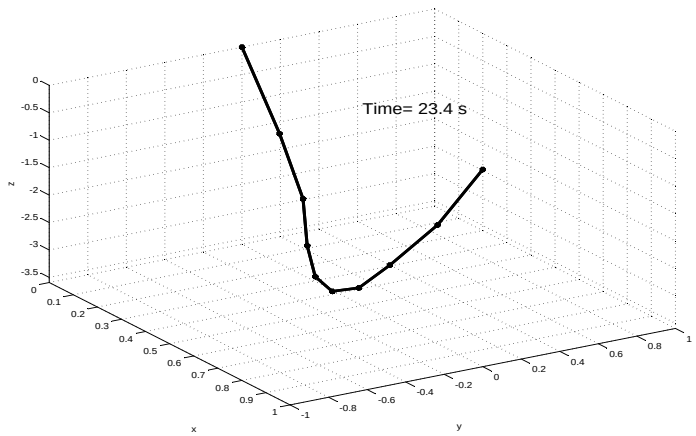

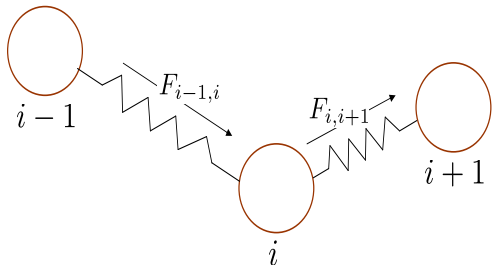

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$$P_{N+1} \longrightarrow P_f$$

- The cost function

$$J(x_0, T_p, \mathbf{u}) := \int_0^{T_p} \Big[\sum_{i=1}^{N_b} \dot{x}_i^2(\tau)\Big] d\tau$$

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Control Parametrization

Use the parameterized explicit feedback :

$$u = Sat_{-u_{max}}^{+u_{max}} \left(-\lambda \left[P_{N+1} - P_{N+1}^{d}(p) \right] \right)$$

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$$P_{N+1}^{d}(p) := P_f + \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} e^{-\lambda_0 t} + \begin{pmatrix} p_4 \\ p_5 \\ p_6 \end{pmatrix} e^{-2\lambda_0 t} + \dots$$

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$\longrightarrow$ The cost function becomes $J(x_0, T_p, p)$

The iteration process $\mathcal{S}$

$$\begin{aligned} p(t_{k+1}) &= \mathcal{S}^{(q)}(p^+(t_k), \hat{x}(t_{k+1})) \\ x(t_{k+1}) &= F(x(t_k), \tau_s, u^1(\hat{p}(x(t_k)))) \end{aligned}$$

At iteration (i),

- Q-approximation of the scalar function $J(p_{k_i})$ together with a trust region updating. $\rightarrow$ compute dp_{k_i}
- Perform the update $p^{(i)} \leftarrow p^{(i)} + dp_{k_i}$ when appropriate
- L-approximation of the scalar function $J(p^{(i)} + \alpha G^{(i)})$ where

$$G^{(i)} = f(dp_1^{(i_1)}, dp_2^{(i_2)}, \ldots, dp_{n_p}^{(i_{n_p})}, \beta)$$

- Perform as many successful steps as possible along along $G^{(i)}$
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- Each Q-approximation : 3 cost function evaluations
- Each L-approximation along the gradient : 1 cost function evaluation

q is the number of allowed cost function evaluations during a sampling period.

Scenario description

1. Sampling period for control $\tau_s = 0.2$ *sec*
2. Platform PC-P4, 3.2GHz, 1Go RAM
3. Initial disturbance :

$$u = Sat_{-u_{max}}^{+u_{max}} \left(-\lambda \left[P_{N+1} - P_{N+1}^d(p) \right] \right) \quad ; \quad P_{N+1}^d(p) := P_f + \begin{pmatrix} 0.5 \\ 4 \\ 4 \end{pmatrix} e^{-\lambda_0 t}$$

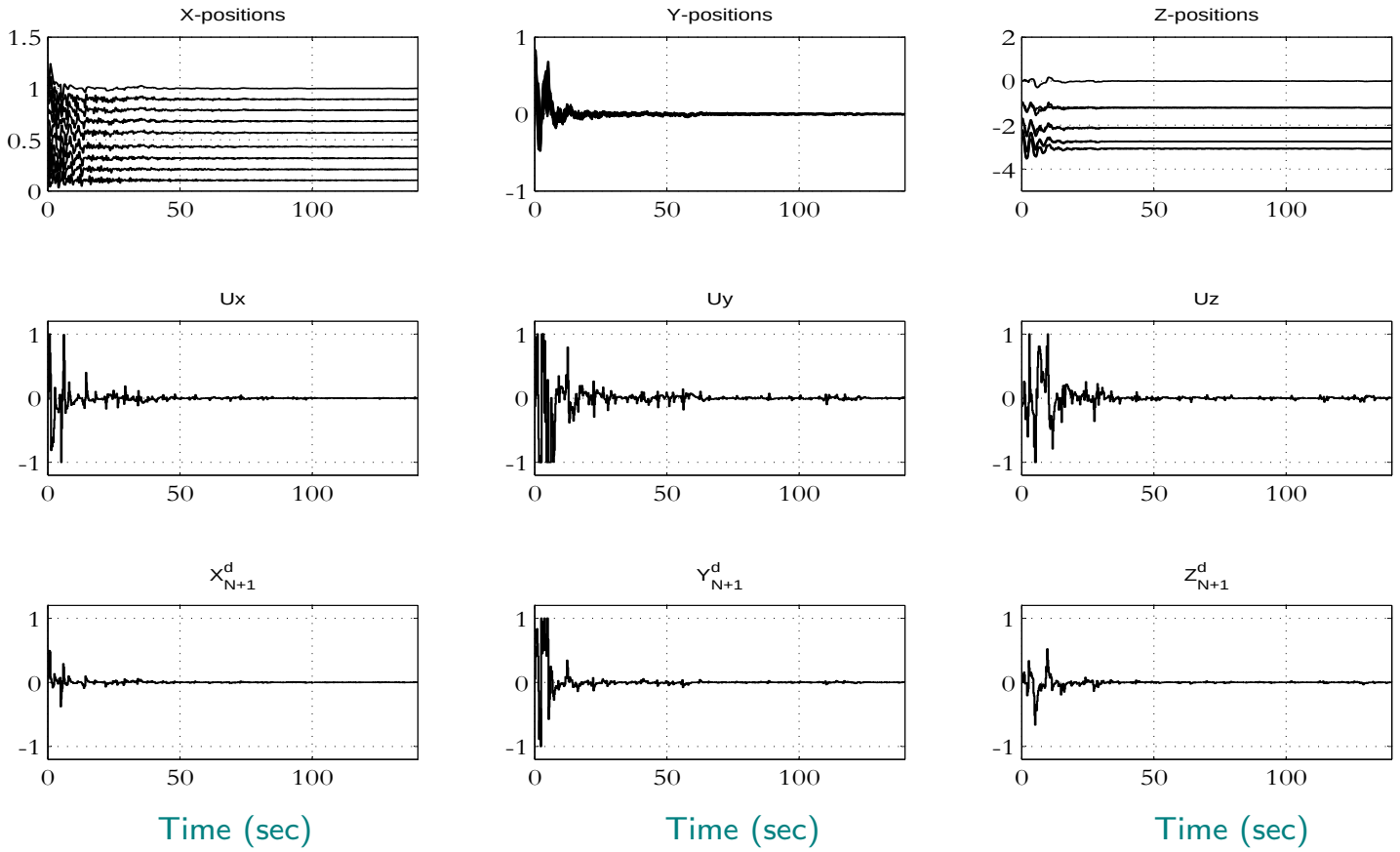
   activation of the controller 5 *sec* later.

4. Prediction horizon $T_p = 5$ *sec* or $T_p = 8$ *sec*.
5. Number or parameters $n_p \in \{3, 6, 9\}$
6. $\lambda = 1$ and $\lambda_0 = 2$
7. Maximal excursion of the end ball $\Delta P_{max} \in \{0.2m, 1m\}$

Ball positions

Control inputs

P^d_{N+1}



$\Delta P_{max} = 1 \quad ; \quad n_p = 3 \quad ; \quad q = 10$

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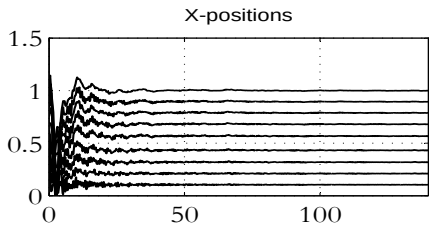
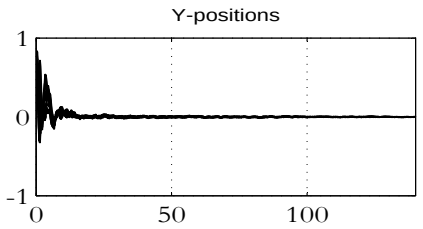
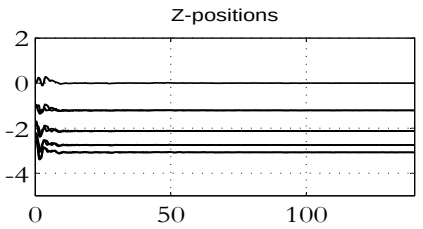
Ball positions







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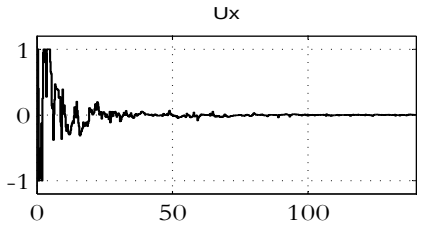
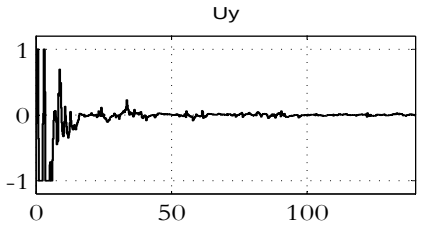
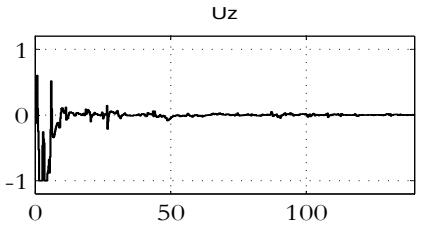
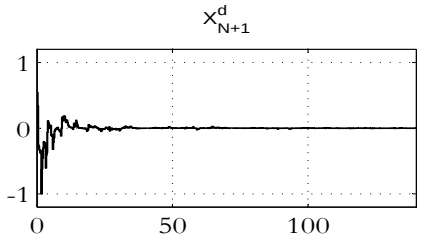
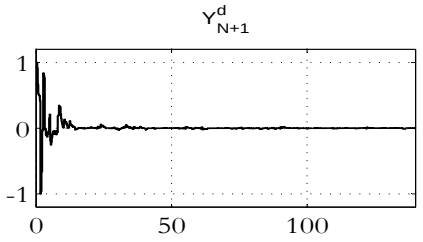


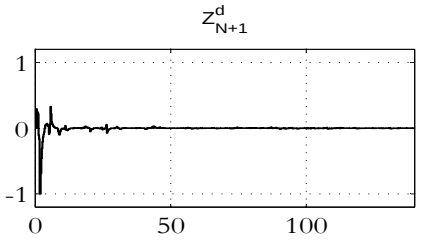




P^d_{N+1}







$\Delta P_{max} = 1 \quad ; \quad n_p = 6 \quad ; \quad q = 10$

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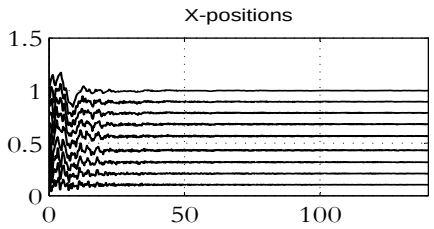
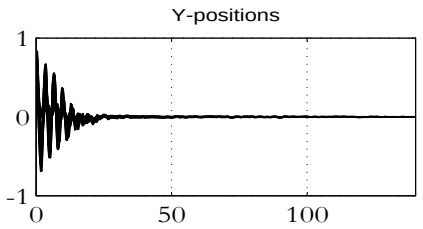
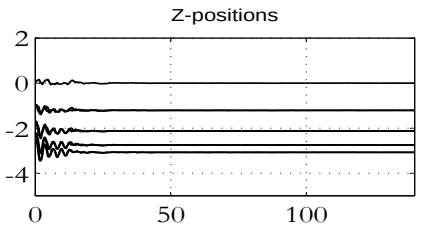
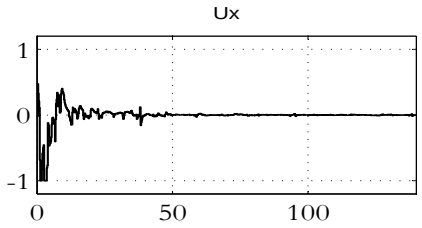
Ball positions

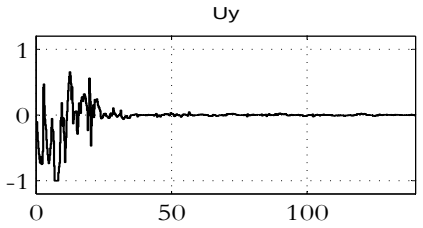


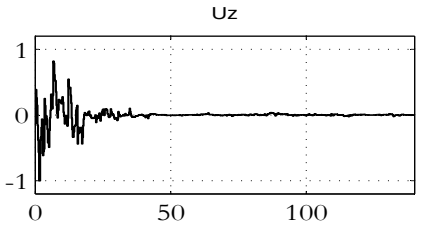




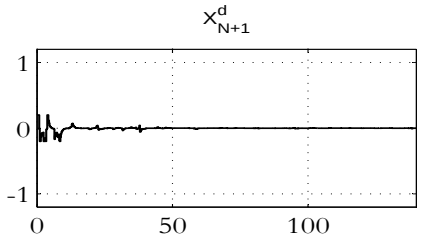
Control inputs

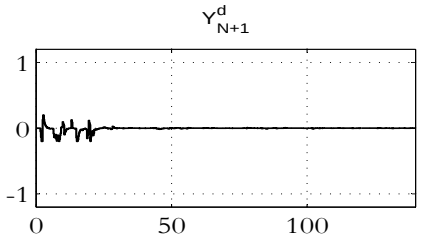


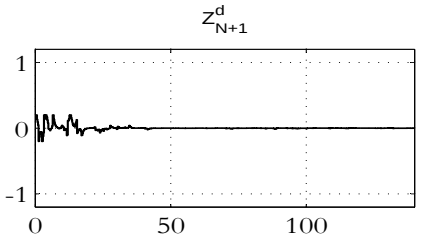




P^d_{N+1}







$\Delta P_{max} = 0.2 \quad ; \quad n_p = 6 \quad ; \quad q = 10$

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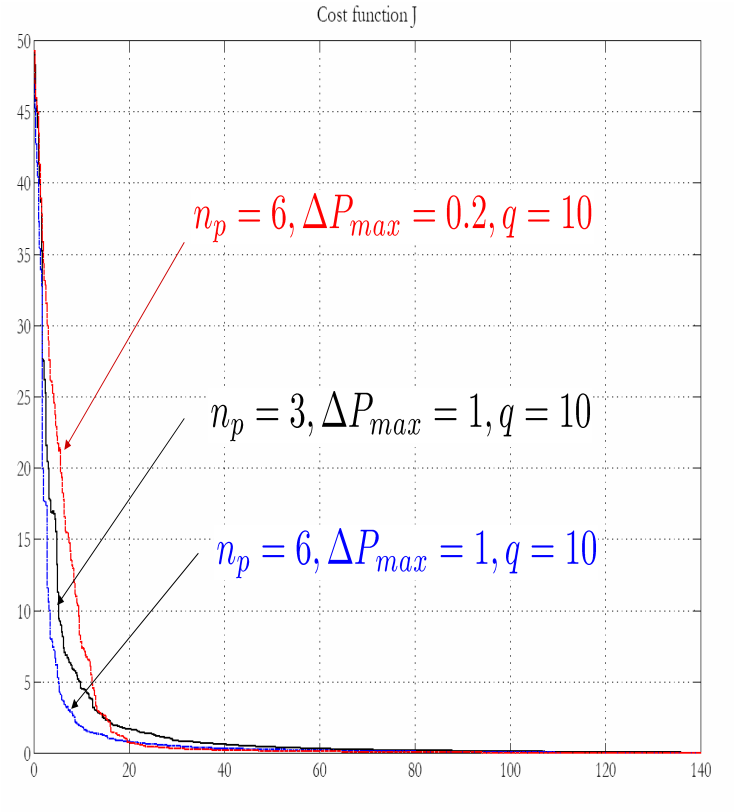
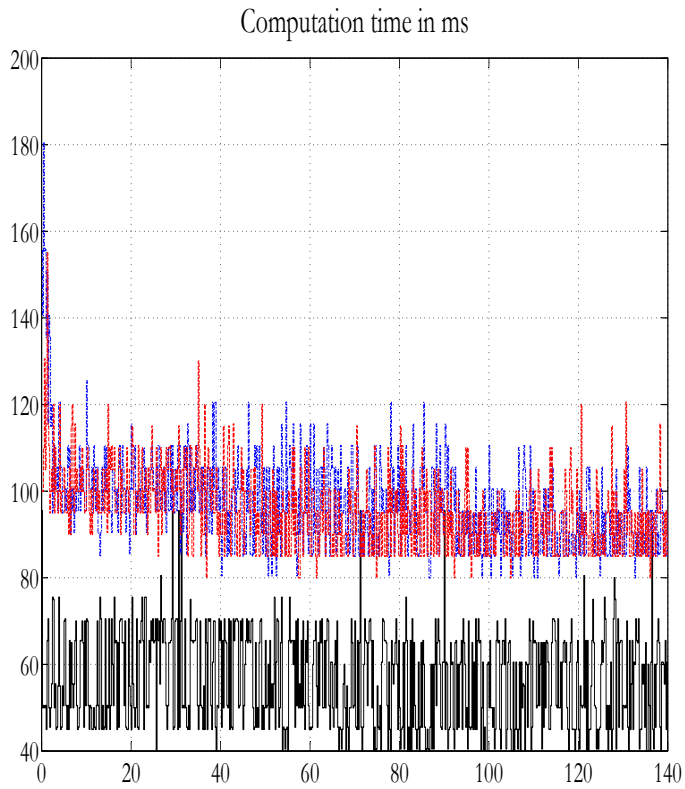
Evolution of $J = \sum_{i=1}^{N_b} \dot{x}_i^2$ in closed-loop / Computation time (ms)





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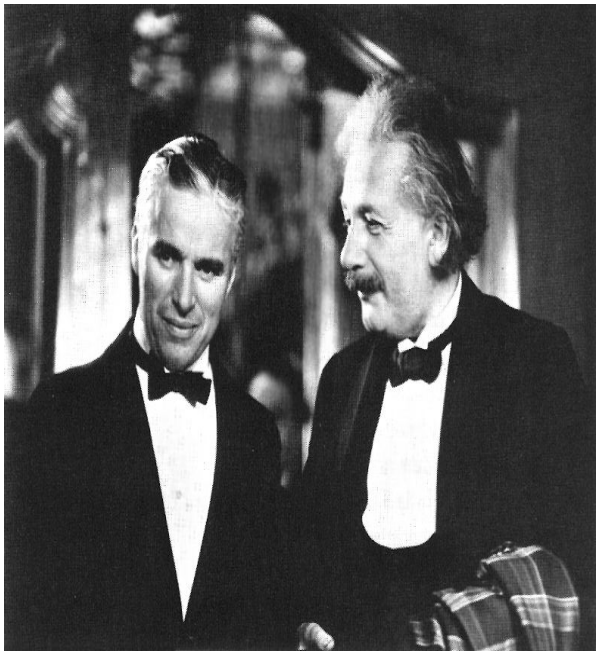
Animations with $n_p = 6, q = 10$

$\Delta P_{max} = 1$



$\Delta P_{max} = 0.2$

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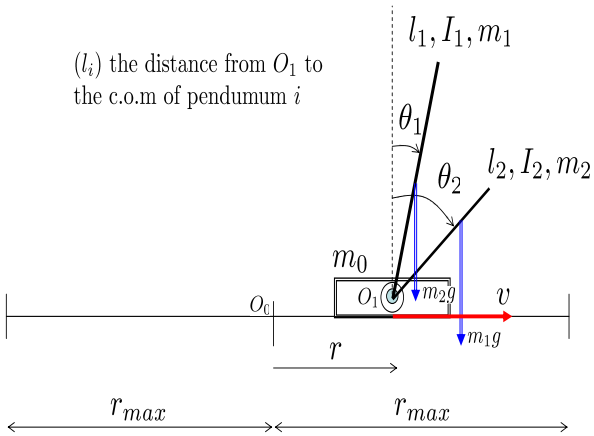
State constrained Swing-up and stabilization of a twin-pendulum system



[M. Alamir & A. Murilo Automatica 2007]

- Constrained State Feedback
- Swing-up & Stabilization
- Saturations on control & State
- Fast Constrained NMPC
- Sampling period 0.1 *sec*

gipsa-lab

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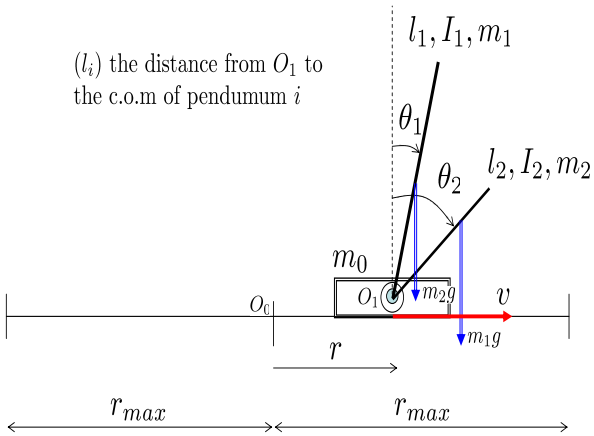

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Preliminary experimental result

- $\dot{T}_j = u$
- $T_j \in [T_j^{min}, T_j^{max}]$
- $u \in [-\bar{u}, +\bar{u}]$

$$p := \begin{pmatrix} \frac{w_1}{\bar{u}} \\ \frac{w_2}{\bar{u}} \\ \frac{t_1}{t_f} \end{pmatrix}^T \in [-1, +1]^2 \times [0, 1]$$